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TRANSVERSE SHEAR CONSIDERATIONS IN

ANISOTROPIC PLATES

THESIS

Richard H. Reams

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TRANSVERSE SHEAR CONSIDERATIONS IN ANISOTROPIC PLATES

THESIS

Presented to the Faculty of the School of Engineering

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a	Plate x dimension
a_{ij}	Normalized extensional stiffness
A_{ij}	Extensional stiffness
A_{mn}	Undetermined admissible function coefficient for ψ_x
b	Plate y dimension
B_{ij}	Extensional - bending coupling stiffness
B_{mn}	Undetermined admissible function coefficient for ψ_y
C_{mn}	Undetermined admissible function coefficient for w
d_{ij}	Normalized bending stiffness
D_{ij}	Bending stiffness
E_i	Elastic modulus in the i direction
E_{ij}	Higher order stiffness
f_{ij}	Normalized Higher Order Stiffness
F_{ij}	Higher order stiffness
G_{ij}	Shear modulus in the i-j plane
h	Plate thickness
h_{ij}	Normalized Higher Order Stiffness
H_{ij}	Higher order stiffness
I_i, $\bar{I}_i$	Mass moments of inertia
k, k_1, k_2	Transverse shear correction factors
k_1,k_2,k_3	Proportional loading coefficients
L	Lagrangian function
m, n	Galerkin algorithm integers (number of terms)
M_i	Moments
N_i	In-plane forces
$\bar{N}_i$	Applied in-plane forces

$\bar{N}_o$	Normalized buckling load
p, q	Galerkin algorithm integers (number of equations)
p	Laminate density thickness product
P_i	Correctional moments
Q_i	Out-of-plane shear forces
Q_{ij}	Reduced stiffnesses
$\bar{Q}_{ij}$	Transformed reduced stiffnesses
R	Plate aspect ratio (a/b); region
R_i	Correctional out-of-plane shear forces
s	Plate span-to-depth ratio (a/h)
S_{ij}	Compliance matrix terms
t	Lamina thickness; time
T	Kinetic energy
U	Strain energy
V	Potential energy due to external forces
u, v, w	Displacements
$\bar{w}$	Normalized out-of-plane displacement
x, y, z	Plate axes
1, 2, 3	Lamina principle axes
δ	First variational operator
ϵ_{1-3}	Normal strains
ϵ_{4-6}	Shear strains
$\varkappa_i$	Curvatures
η	Normalized y coordinate
θ	Fiber orientation angle
ν_{ij}	Poisson's ratio in the i-j plane
ξ	Normalized x coordinate

ξ_i	Undetermined function
π	3.1415927
ρ	Density
σ_{1-3}	Normal stresses
σ_{4-6}	Shear stresses
ψ_i	Rotations due to bending
ω	Natural frequency
$\bar{\omega}$	Normalized natural frequency
ζ_i	Undetermined function

Abstract

This work was initiated by a need for solutions to a higher order shear deformation plate theory, which can better approximate the through-the-thickness deformation and interlaminar shear stresses, using plates with boundary conditions other than simply supported. The non-simply supported boundary conditons necessitated the use of an approximate technique, namely, the Galerkin technique. Solutions desired were natural frequencies and buckling loads. Boundary conditions considered were: simply supported, clamped, and clamped - simply supported. Two graphite/epoxy laminates of different construction were investigated for effects of varying span-to-depth ratios. Comparisons of the higher order theory with linear shear theory and classical laminated plate theory were made along with convergence characteristics of the Galerkin technique.

I. Introduction

Background

Laminated composite materials are now commonplace on aerospace vehicles. The high specific strength and stiffness of composites is ideally suited to weight critical aerospace structures. The tailorability of composites gives the designer choices previously unavailible with isotropic materials.

Classical laminated plate theory does not accurately predict deflections and natural frequencies of thick composite plates. A part of the problem lies in the fact that the classical laminated plate theory omits transverse shear strains. This results in an underprediction of deflections and overprediction of natural frequencies and buckling loads. These errors are further compounded by the high ratios of elastic modulus to shear modulus of most composite materials.

The inclusion of transverse shear strains into plate theories have been performed by many authors [1-5]. These theories use displacement fields which account for a linear or higher order variation of midplane displacement through the thickness. Mindlin's theory [1] is a two dimensional linear approach wherein the change in displacement is a result of a rotation due to bending and a rotation due to shear deformation. Reissner introduced a similar theory [2]. The aforementioned theories violate conditions of zero transverse shear stress on the upper and lower surfaces of the plate. The theories also assume zero warping of the cross section due to shear and require corrections to the transverse shear stiffnesses. In general, three dimensional theories are complex to apply, such as those given by [6]

and [7].

Several higher order theories have been developed to include a parabolic variation of the transverse shear strain through the thickness. With such a theory, no shear correction coefficients are needed. Theories presented by Levinson [8] and Murthy [9] use different displacement fields, but both use the first order shear displacement theory equilibrium equations. This approach is variationally inconsistent. A theory which uses the same displacement field as Levinson but contains equilibrium equations based on the principle of virtual displacements was proposed by Reddy [10].

Reddy's other related papers [11] and [12] consider stability and natural vibration of laminated plates using the finite element method and the Navier solution procedure. These papers only consider laminated plates with boundaries that are simply supported.

Therefore, an alternate method is required to obtain solutions for plates with boundary conditions other than simply supported using Reddy's variationally consistent higher order shear displacement theory.

Objectives

This thesis will apply a higher order shear deformation theory to two different symmetrically laminated square plates. Natural frequencies and buckling loads will be found for the following three boundary conditions: simply supported on all four edges; clamped on all four edges; simply supported on two opposite sides, clamped on the other two sides. Various span-to-depth ratios will be chosen to investigate their effects. Results from the higher order theory wil be compared with linear shear deformation theory which will incorporate one transverse shear stiffness correction coefficient and rotatory inertia. Comparisons will also be made with findings from classical laminated plate theory.

Approach

This thesis will use the higher order shear deformation theory developed by Reddy. Complete equations of motion will be derived using Hamilton's principle beginning with Reddy's displacement field. The Galerkin technique will be applied to the resulting differential equations for the three boundary conditions mentioned in the objectives. Admissible functions for each boundary condition will be chosen and the Galerkin procedure will be carried out. The resulting equations will be programmed into a Galerkin algorithm to generate the stiffness and mass matricies. An eigenvalue problem will result for each boundary condition and will be solved using another computer program. Natural frequencies and buckling loads will be products of this program. Convergence characteristics of the Galerkin solution technique are to be investigated.

II. Theory and Modeling

The theory intrinsic to this thesis will begin with introductory material on the constitutive equations governing laminated plates. Following that development, the kinematics of the higher order plate theory will be introduced. Variations from linear theory kinematics will be noted. Next, Hamilton's principle will be used to derive the equations of motion. The equations of motion will then be normalized and the Galerkin approximate solution technique invoked.

Constitutive Equations

Because of the inherent anisotropy of composite materials, a coordinate system must be defined to give a reference between fiber direction and plate axes. The x and y directions will denote plate axes, while the 1 direction will lie along the fiber axis and the 2 direction will lie transverse to the fibers as seen in Figure 2.1.

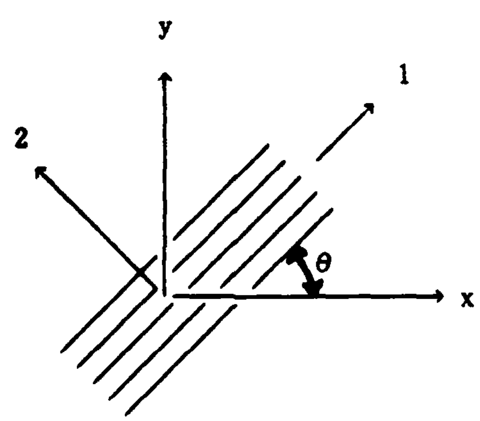

Figure 2.1 Coordinate System Definition

With the above defined coodinate system, the constitutive equations may be written as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \qquad (1)$$

where the 1, 2, and 3 subscripts denote normal stresses and strains and the 4, 5, and 6 subscripts denote shear stresses and strains. The S_{ij} matrix is referred to as the compliance matrix. The development of the compliance matrix may be found in [13]. Eq.(1) can be inverted to give

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{13} & 0 & 0 & 0 \\ Q'_{12} & Q'_{22} & Q'_{23} & 0 & 0 & 0 \\ Q'_{13} & Q'_{23} & Q'_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q'_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \qquad (2)$$

where Q'_{ij} are the reduced stiffnesses. Relations between $[S]$ and $[Q']$ can also be found in [13].

In order to work in the plate axes, the stiffnesses must be resolved into x and y components. The stiffness matrix must undergo a

tensor transformation as follows

$$\left[\bar{Q}'\right] = \left[T\right]\left[Q'\right]\left[T\right]^T \tag{3}$$

where

$$\left[T\right] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix} \tag{4}$$

and $m=\cos\theta$ and $n=\sin\theta$. Thus, the constitutive relation in plate axes is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \bar{Q}'_{11} & \bar{Q}'_{12} & \bar{Q}'_{13} & 0 & 0 & \bar{Q}'_{16} \\ \bar{Q}'_{12} & \bar{Q}'_{22} & \bar{Q}'_{23} & 0 & 0 & \bar{Q}'_{26} \\ \bar{Q}'_{13} & \bar{Q}'_{23} & \bar{Q}'_{33} & 0 & 0 & \bar{Q}'_{36} \\ 0 & 0 & 0 & \bar{Q}'_{44} & \bar{Q}'_{45} & 0 \\ 0 & 0 & 0 & \bar{Q}'_{45} & \bar{Q}'_{55} & 0 \\ \bar{Q}'_{16} & \bar{Q}'_{26} & \bar{Q}'_{36} & 0 & 0 & \bar{Q}'_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \tag{5}$$

For thin laminates, an assumption of plane stress is valid. A plane stress state exists when σ_3 is small compared to σ_1 and σ_2. Thus, we will set $\sigma_3=0$ and remove it from Eq.(5). Therefore

$$\sigma_3 = 0 = \bar{Q}'_{13}\varepsilon_1 + \bar{Q}'_{23}\varepsilon_2 + \bar{Q}'_{33}\varepsilon_3 + \bar{Q}'_{36}\varepsilon_6 \tag{6}$$

Strain through the thickness, ε_3, may be found by rearranging Eq.(6) to

yield

$$\varepsilon_3 = (\bar{Q}'_{13} / \bar{Q}'_{33})\varepsilon_1 + (\bar{Q}'_{23} / \bar{Q}'_{33})\varepsilon_2 + (\bar{Q}'_{36} / \bar{Q}'_{33})\varepsilon_6 \tag{7}$$

Because of plane stress, $\left[\bar{Q}'\right]$ is now a 5 X 5 matrix and is called $\left[\bar{Q}\right]$, the transformed reduced stiffness matrix, which is

$$\left[\bar{Q}\right] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \tag{8}$$

where

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta) \\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta \\
\bar{Q}_{44} &= Q_{44}\cos^2\theta + Q_{55}\sin^2\theta \\
\bar{Q}_{45} &= (Q_{44}-Q_{55})\cos\theta\sin\theta \\
\bar{Q}_{55} &= Q_{55}\cos^2\theta + Q_{44}\sin^2\theta \\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta)
\end{aligned} \tag{9}$$

and

$$Q_{11} = E_1 / (1-\nu_{12}\nu_{21})$$

$$Q_{12} = \nu_{12}E_2 / (1-\nu_{12}\nu_{21}) = \nu_{21}E_1 / (1-\nu_{12}\nu_{21})$$

$$Q_{22} = E_2 / (1-\nu_{12}\nu_{21}) \tag{10}$$

$$Q_{44} = G_{23}$$

$$Q_{55} = G_{31}$$

$$Q_{66} = G_{12}$$

where E_1 is the longitudinal modulus, E_2 the transverse modulus, ν_{12} and ν_{21} are Poisson's ratios, and G_{23}, G_{31}, G_{12} are the shear moduli.

The constitutive equations in final form are therefore

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \tag{11}$$

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{12}$$

where the transverse shear terms σ_4 and σ_5 have been separated out from the inplane components.

Kinematics

The kinematics to be used in this thesis are those developed by Reddy [10]. Reddy's approach was to chose a displacement field with special properties that would satisfy conditions of the transverse shear

stresses being zero on the surface and non-zero through the thickness. A displacement field that meets those conditions is

$$
\begin{aligned}
u_1(x,y,z,t) &= u_o(x,y,t) + z\psi_x(x,y,t) + z^2\xi_x(x,y,t) + z^3\zeta_x(x,y,t) \\
u_2(x,y,z,t) &= v_o(x,y,t) + z\psi_y(x,y,t) + z^2\xi_y(x,y,t) + z^3\zeta_y(x,y,t) \\
u_3(x,y,t) &= w(x,y,t)
\end{aligned}
\tag{13}
$$

where u_o, v_o, and w are displacements of the plate's midplane at a point (x,y). u_o and v_o are inplane displacements parallel to the x and y axes respectively. Out of plane displacements are given by w. ψ_x and ψ_y are bending rotations of the normals to the midplane about the y and x axes respectively. ξ_x, ξ_y, ζ_x, and ζ_y are undetermined functions and all variables are functions of time. Functions ξ_x, ξ_y, ζ_x, and ζ_y are determined by satisfying zero transverse shear stresses at the top and bottom surfaces, i.e.

$$
\sigma_5(x,y,\pm h/2,t) = 0 \qquad ; \qquad \sigma_4(x,y,\pm h/2,t) = 0 \tag{14}
$$

The transverse shear strains are, using Eq.(13) and infinitesimal linear strain theory

$$
\begin{aligned}
\varepsilon_5 &= u_{1},_{z} + u_{3},_{x} = \psi_x + 2z\xi_x + 3z^2\zeta_x + w,_x \\
\varepsilon_4 &= u_{2},_{z} + u_{3},_{y} = \psi_y + 2z\xi_y + 3z^2\zeta_y + w,_y
\end{aligned}
\tag{15}
$$

where the comma denotes partial differentiation. Setting $\sigma_4 = \sigma_5 = 0$ on the plate surface in the transverse shear constitutive equation

(Eq.(12)) we obtain

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{16}$$

also

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{17}$$

or

$$\varepsilon_4 = (-\bar{Q}_{45} / \bar{Q}_{44}) \, \varepsilon_5 \qquad ; \qquad \varepsilon_5 = (-\bar{Q}_{45} / \bar{Q}_{55}) \, \varepsilon_4 \tag{18}$$

For orthotropic plates $\bar{Q}_{45} = 0$, and the strains are automatically zero. Otherwise, $\bar{Q}_{45}$ is nonzero. For non-orthotropic plates we will set $\varepsilon_4 = \varepsilon_5 = 0$ at the top and bottom surfaces because there is no traction applied there. Thus, with $\varepsilon_5(x,y,\pm h/2) = \varepsilon_4(x,y,\pm h/2) = 0$ we have from Eq.(15)

$$\xi_x = 0 \qquad ; \qquad \xi_y = 0$$

$$\zeta_x = -4/3h^2 \, (w,_x + \psi_x) \qquad ; \qquad \zeta_y = -4/3h^2 \, (w,_y + \psi_y) \tag{19}$$

Substituting Eq.(19) into our original displacement field Eq.(13), the desired displacement field is obtained. Thus

$$u_1 = u_o + z \left[\psi_x - 4/3 \; (z/h)^2 \; (\psi_x + w,_x) \right]$$

$$u_2 = v_o + z \left[\psi_y - 4/3 \; (z/h)^2 \; (\psi_y + w,_y) \right] \qquad (20)$$

$$u_3 = w$$

This displacement field resembles those given for other transverse shear theories. In most linear theories, the second term within the square brackets does not appear. However, the dependent variables are the same as the linear theories. Compared with other higher order theories, the displacement field is identical to Levinson's [8] (with the exception of inplane displacements) but different from Murthy's [9].

Using infinitesimal linear strain-displacement relations and Eq.(20), we find the strains to be

$$\begin{aligned} \varepsilon_1 &= \varepsilon_1^0 + z \; (\kappa_1^0 + z^2 \kappa_1^2) \\ \varepsilon_2 &= \varepsilon_2^0 + z \; (\kappa_2^0 + z^2 \kappa_2^2) \\ \varepsilon_3 &= 0 \\ \varepsilon_4 &= \varepsilon_4^0 + z^2 \kappa_4^2 \\ \varepsilon_5 &= \varepsilon_5^0 + z^2 \kappa_5^2 \\ \varepsilon_6 &= \varepsilon_6^0 + z \; (\kappa_6^0 + z^2 \kappa_6^2) \end{aligned} \qquad (21)$$

where

$$\varepsilon_1^0 = u_0{}',_x \quad ; \quad \kappa_1^0 = \psi_x{}',_x \quad ; \quad \kappa_1^2 = -4/_{3h^2} \; (\psi_x{}',_x + w,_{xx})$$

$$\varepsilon_2^0 = v_0{}',_y \quad ; \quad \kappa_2^0 = \psi_y{}',_y \quad ; \quad \kappa_2^2 = -4/_{3h^2} \; (\psi_y{}',_y + w,_{yy})$$

$$\varepsilon_4^0 = \psi_y + w,_y \quad ; \quad \kappa_4^2 = -4/_{h^2}\,(\psi_y + w,_y) \tag{22}$$

$$\varepsilon_5^0 = \psi_x + w,_x \quad ; \quad \kappa_5^2 = -4/_{h^2}\,(\psi_x + w,_x)$$

$$\varepsilon_6^0 = u_0,_y + v_0,_x \quad ; \quad \kappa_6^0 = \psi_x,_y + \psi_y,_x$$

$$\kappa_6^2 = -4/_{3h^2}\,(\psi_x,_y + \psi_y,_x + 2w,_{xy})$$

The transverse shear strains are seen to be

$$\varepsilon_4 = \psi_y + w,_y + z^2(-4/h^2)(\psi_y + w,_y) = (\underline{1-4/h^2\,z^2})\,(\psi_y + w,_y) = \gamma_{yz}$$

$$\varepsilon_5 = \psi_x + w,_x + z^2(-4/h^2)(\psi_x + w,_x) = (\underline{1-4/h^2\,z^2})\,(\psi_x + w,_x) = \gamma_{xz} \tag{23}$$

The underlined quantities indicate terms not present in linear shear deformation theory. Eqs.(23) illustrate the parabolic variation of the shear strains through the thickness of the plate. The strains are seen to be zero on the bottom and top surfaces of the plate ($z=-h/2$ and $z=h/2$ respectively). Figure 2.2 shows the relationship between ψ_x, $w,_x$, and γ_{xz} for the linear theory.

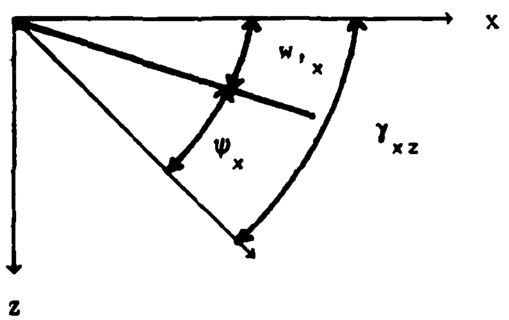

Figure 2.2 Shear Deformation Definition

The higher theory can be shown to degenerate to the linear theory. From [10], the average shear force per unit length on an edge perpendicular to the x axis is shown as

$$(Q_5)_{avg} = {}^1/_2 \int_{-h/2}^{h/2} \sigma_5 \, dz \tag{24}$$

Using Eqs.(12) and (23) in Eq.(24) we obtain

$$(Q_5)_{avg} = {}^1/_2 \int_{-h/2}^{h/2} (1-4/h^2 z^2)dz \left[\bar{Q}_{45}(\psi_x + w,_x) + \bar{Q}_{55}(\psi_x + w,_x) \right] \tag{25}$$

After integration Eq.(25) becomes

$$(Q_5)_{avg} = 5/6 \; h \left[\bar{Q}_{45}(\psi_x + w,_x) + \bar{Q}_{55}(\psi_x + w,_x) \right] \tag{26}$$

or

$$(Q_5)_{avg} = 5/6 \left[A_{45}(\psi_x + w,_x) + A_{55}(\psi_x + w,_x) \right] \tag{27}$$

Thus a transverse shear correction coefficient of 5/6 results. This coefficient is used to modify the stiffness terms to account for zero warping of the midplane normals. The higher order theory gives a parabolic variation through the thickness which allows warping and thus does not require a correction coefficient.

Equations of Motion

The equations of motion will now be developed in a variationally consistent manner using Hamilton's principle. Hamilton's principle states that "among all dynamic paths that satisfy the boundary conditions over the boundary surface at all times and that start and end with the actual values at two arbitrary instants of time t_1 and t_2 at every point in the body, the "actual" dynamic path is distinguished by making the Lagrangian function an extremum." [14] The Lagrangian function is given as

$$L = T - V - U \tag{28}$$

Hamilton's principle is therefore

$$\delta \int_{t_1}^{t_2} (T - V - U) \, dt = 0 \tag{29}$$

where T is the kinetic energy, V is the potential energy due to external forces, and U is the strain energy. The δ symbol denotes the first variation. For rigid body dynamics U is zero and for statics T is zero. Hamilton's principle (Eq.(29)) may be rewritten to give

$$\int_{t_1}^{t_2} (\delta T - \delta V - \delta U) \, dt = 0 \tag{30}$$

Derivations for the first variation of kinetic energy, potential energy

due to external forces, and strain energy are given separately in the following subparagraphs.

Kinetic Energy The kinetic energy is given in [15] as

$$T = {}^{1}/_{2} \iiint_R \int_{-h/2}^{h/2} \rho \, \dot{u}_i \, \dot{u}_i \, dz \, dx \, dy \tag{31}$$

where the $\dot{}$ symbol indicates a derivative with respect to time and ρ is the material density. Rewriting Eq.(31)

$$T = {}^{1}/_{2} \iiint_R \int_{-h/2}^{h/2} (\rho \, (\dot{u}_1)^2 + \rho \, (\dot{u}_2)^2 + \rho \, (\dot{u}_3)^2) \, dz \, dx \, dy \tag{32}$$

After substituting our assumed displacement field Eq.(20) and defining the plate inertias to be

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \int_{-h/2}^{h/2} \rho \, (1, z, z^2, z^3, z^4, z^6) \, dz$$

$$\bar{I}_2 = I_2 - (4/3h^2) \, I_4 \quad ; \quad \bar{I}_5 = I_5 - (4/3h^2) \, I_7 \tag{33}$$

$$\bar{I}_3 = I_3 - (8/3h^2) \, I_5 + (16/9h^4) \, I_7$$

Eq.(32) becomes

$$T = {}^1/_2 \iint_R \Big\{ I_1\dot{u}_o^2 + \bar{I}_2\dot{\psi}_x\dot{u}_o + \bar{I}_3\dot{\psi}_x^2 - 2(4/3h^2)I_4\dot{w},_x\dot{u}_o - 2(4/3h^2)\bar{I}_5\dot{w},_x\dot{\psi}_x$$

$$(4/3h^2)^2 I_7\dot{w},_x^2 + I_1\dot{v}_o^2 + 2\bar{I}_2\dot{\psi}_y\dot{v}_o + \bar{I}_3\dot{\psi}_y^2 - 2(4/3h^2)I_4\dot{w},_y\dot{v}_o$$

$$- 2(4/3h^2)\bar{I}_5\dot{w},_y\dot{\psi}_y + (4/3h^2)^2 I_7\dot{w},_y^2 + I_1\dot{w}^2 \Big\}\, dxdy \qquad (34)$$

If one carrys out the variation operation, the result is

$$\delta T = \iint_R \Big\{ \left(I_1\dot{u}_o + \bar{I}_2\dot{\psi}_x - (4/3h^2)I_4\dot{w},_x \right) \delta\dot{u}_o$$

$$+ \left(I_1\dot{v}_o + \bar{I}_2\dot{\psi}_y - (4/3h^2)I_4\dot{w},_y \right) \delta\dot{v}_o$$

$$+ \left(-(4/3h^2)I_4\dot{u}_o - (4/3h^2)\bar{I}_5\dot{\psi}_x + (4/3h^2)^2 I_7\dot{w},_x \right) \delta\dot{w},_x$$

$$+ \left(-(4/3h^2)I_4\dot{v}_o - (4/3h^2)\bar{I}_5\dot{\psi}_y + (4/3h^2)^2 I_7\dot{w},_y \right) \delta\dot{w},_y$$

$$+ \left(\bar{I}_2\dot{u}_o + \bar{I}_3\dot{\psi}_x - (4/3h^2)\bar{I}_5\dot{w},_x \right) \delta\dot{\psi}_x$$

$$+ \left(\bar{I}_2\dot{v}_o + \bar{I}_3\dot{\psi}_y - (4/3h^2)\bar{I}_5\dot{w},_y \right) \delta\dot{\psi}_y + \left(I_1\dot{w} \right) \delta\dot{w} \Big\}\, dx\, dy \qquad (35)$$

We now integrate Eq.(35) by parts and choose a plate coordinate system as shown in Figure 2.3

$$\delta T = \int_0^b \int_0^a \Big\{ \left(I_1\dot{u}_o + \bar{I}_2\dot{\psi}_x - (4/3h^2)I_4\dot{w},_x \right) \delta\dot{u}_o$$

$$+ \left(I_1\dot{v}_o + \bar{I}_2\dot{\psi}_y - (4/3h^2)I_4\dot{w},_y \right) \delta\dot{v}_o$$

$$+ \Big((4/3h^2) I_4 \dot{u}_{o,x} + (4/3h^2) \bar{I}_5 \dot{\psi}_{x,x} - (4/3h^2)^2 I_7 \dot{w}_{,xx} + (4/3h^2) I_4 v_{o,y}$$

$$+ (4/3h^2) \bar{I}_5 \dot{\psi}_{y,y} - (4/3h^2)^2 I_7 \dot{w}_{,yy} + I_1 \dot{w} \Big) \delta\dot{w}$$

$$\Big(\bar{I}_2 u_o + \bar{I}_3 \dot{\psi}_x - (4/3h^2) \bar{I}_5 \dot{w}_{,x} \Big) \delta\dot{\psi}_x$$

$$+ \Big(\bar{I}_2 \dot{v}_o + \bar{I}_3 \dot{\psi}_y - (4/3h^2) \bar{I}_5 \dot{w}_{,y} \Big) \delta\dot{\psi}_y \Big\} dxdy$$

$$+ \int_0^b \Big\{ -(4/3h^2) I_4 \dot{u}_o - (4/3h^2) \bar{I}_5 \dot{\psi}_x + (4/3h^2)^2 I_7 \dot{w}_{,x} \Big\} \delta\dot{w} \Big|_0^a dy$$

$$+ \int_0^a \Big\{ (4/3h^2) I_4 \dot{v}_o + (4/3h^2) \bar{I}_5 \dot{\psi}_y - (4/3h^2)^2 I_7 \dot{w}_{,y} \Big\} \delta\dot{w} \Big|_b^0 dx \qquad (36)$$

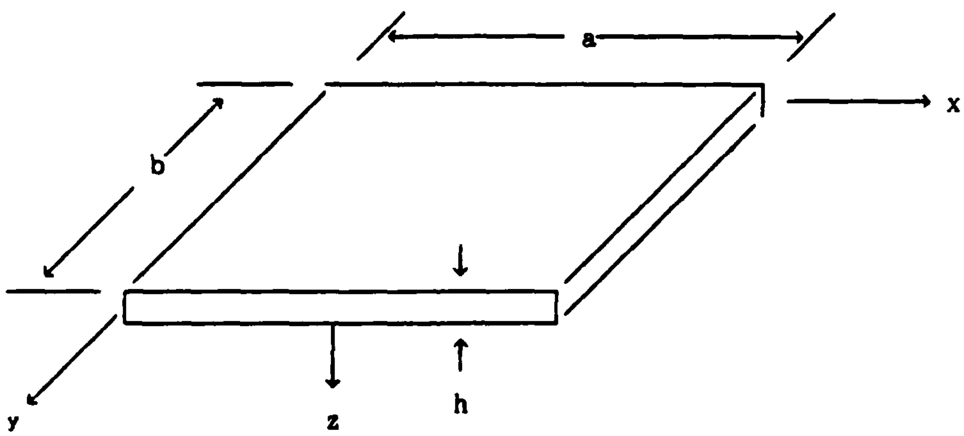

Figure 2.3 Plate Coordinate System

Because we will not deal with time dependant boundaries, the last two integrals of Eq.(36) may be eliminated. Thus

$$\delta T = \int_0^b \int_0^a \Big\{ \Big(I_1 \dot{u}_o + \bar{I}_2 \dot{\psi}_x - (4/3h^2) I_4 \dot{w}_{,x} \Big) \delta\dot{u}_o$$

$$+ \left(I_1 \dot{v}_o + \bar{I}_2 \dot{\psi}_y - (4/3h^2) I_4 \dot{w},_y \right) \delta \dot{v}_o$$

$$+ \left((4/3h^2) I_4 \dot{u}_{o,x} + (4/3h^2) \bar{I}_5 \dot{\psi}_{x,x} - (4/3h^2)^2 I_7 \dot{w},_{xx} + (4/3h^2) I_4 v_{o,y} \right.$$

$$\left. + (4/3h^2) \bar{I}_5 \dot{\psi}_{y,y} - (4/3h^2)^2 I_7 \dot{w},_{yy} + I_1 \dot{w} \right) \delta \dot{w}$$

$$\left(\bar{I}_2 u_o + \bar{I}_3 \dot{\psi}_x - (4/3h^2) \bar{I}_5 \dot{w},_x \right) \delta \dot{\psi}_x$$

$$\left. + \left(\bar{I}_2 \dot{v}_o + \bar{I}_3 \dot{\psi}_y - (4/3h^2) \bar{I}_5 \dot{w},_y \right) \delta \dot{\psi}_y \right\} dxdy \qquad (37)$$

<u>Potential</u> <u>Energy</u> The potential energy due to external forces is given by

$$V = {}^1/_2 \iint_R \left\{ \bar{N}_1 (w,_x)^2 + \bar{N}_2 (w,_y)^2 + 2\, \bar{N}_6\, w,_x w,_y \right\} dxdy \qquad (38)$$

where $\bar{N}_1$, $\bar{N}_2$, and $\bar{N}_6$ are inplane compression and shear forces. A uniform lateral load q will not be considered. Taking the first variation of Eq.(38) we have

$$\delta V = {}^1/_2 \iint_R \left\{ (2\bar{N}_1 w,_x + 2\bar{N}_6 w,_y) \delta w,_x + (2\bar{N}_2 w,_y + 2\bar{N}_6 w,_x) \delta w,_y \right\} dxdy \qquad (39)$$

Integrating by parts using

$$\iint_R \frac{\partial F}{\partial w,_x} \delta w,_x \, dxdy = -\iint_R \delta w \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial w,_x} \right) dxdy + \int_\Gamma n_j \frac{\partial F}{\partial w,_x} \delta w \, d\Gamma \qquad (40)$$

where Γ represents the boundary and n_j is the component normal to the boundary, Eq.(39) becomes

$$\delta V = \iint_R \left\{ - \bar{N}_1 w,_{xx} - \bar{N}_6 w,_{xy} - \bar{N}_2 w,_{yy} - \bar{N}_6 w,_{xy} \right\} \delta w \; dxdy$$

$$+ \int_\Gamma \left\{ \bar{N}_1 w,_y + \bar{N}_6 w,_y \right\} \delta w \; dy + \int_\Gamma \left\{ - \bar{N}_2 w,_y - \bar{N}_6 w,_x \right\} \delta w \; dx \qquad (41)$$

Using the plate dimensions of Figure 2.3 and expanding the boundary integrals by marching counterclockwise around the plate (starting at the origin and progressing down the x axis), we obtain

$$\delta V = \int_0^b \int_0^a \left\{ -\bar{N}_1 w,_{xx} - \bar{N}_2 w,_{yy} - 2 \, \bar{N}_6 w,_{xy} \right\} \delta w \; dxdy$$

$$+ \int_0^b \left\{ \bar{N}_1 w,_x + \bar{N}_6 w,_y \right\} \Big|_0^a \delta w \; dy + \int_0^a \left\{ - \bar{N}_2 w,_y - \bar{N}_6 w,_x \right\} \Big|_b^0 \delta w \; dx \qquad (42)$$

<u>Strain Energy</u> For a linear elastic material in a state of plane stress, the strain energy may be written as

$$U = {}^1/_2 \iiint_R \int_{-h/2}^{h/2} (\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_4 \varepsilon_4 + \sigma_5 \varepsilon_5 + \sigma_6 \varepsilon_6) \; dzdxdy \qquad (43)$$

With $\sigma_i = E \, \varepsilon_i$ we have

$$U = {}^{1}/_{2} \iiint_{R} \int_{-h/2}^{h/2} (E\varepsilon_1^2 + E\varepsilon_2^2 + E\varepsilon_4^2 + E\varepsilon_5^2 + E\varepsilon_6^2)\, dzdxdy \qquad (44)$$

Performing the first variation operation, one obtains

$$\delta U = \iiint_{R} \int_{-h/2}^{h/2} (E\varepsilon_1 \delta\varepsilon_1 + E\varepsilon_2 \delta\varepsilon_2 + E\varepsilon_4 \delta\varepsilon_4 + E\varepsilon_5 \delta\varepsilon_5 + E\varepsilon_6 \delta\varepsilon_6)\, dzdxdy \quad (45)$$

or

$$\delta U = \iiint_{R} \int_{-h/2}^{h/2} (\sigma_1 \delta\varepsilon_1 + \sigma_2 \delta\varepsilon_2 + \sigma_4 \delta\varepsilon_4 + \sigma_5 \delta\varepsilon_5 + \sigma_6 \delta\varepsilon_6)\, dzdxdy \qquad (46)$$

Substituting strains from Eq.(21) into Eq.(46) we obtain

$$\delta U = \iiint_{R} \int_{-h/2}^{h/2} \Big\{ \sigma_1 \delta(\varepsilon_1^{\circ} + z(\kappa_1^{\circ} + z^2 \kappa_1^2)) + \sigma_2 \delta(\varepsilon_2 + z(\kappa_2^{\circ} + z^2 \kappa_2^2))$$

$$+ \sigma_4 \delta(\varepsilon_4^{\circ} + z^2 \kappa_4^2) + \sigma_5 \delta(\varepsilon_5^{\circ} + z^2 \kappa_5^2) + \sigma_6 \delta(\varepsilon_6^{\circ} + z(\kappa_6^{\circ} + z^2 \kappa_6^2)) \Big\}\, dzdxdy \quad (47)$$

We next integrate Eq.(47) with respect to z

$$\delta U = \iint_{R} \Big\{ N_1 \delta\varepsilon_1^{\circ} + M_1 \delta\kappa_1^{\circ} + P_1 \delta\kappa_1^2 + N_2 \delta\varepsilon_2^{\circ} + M_2 \delta\kappa_2^{\circ} + P_2 \delta\kappa_2^2 + Q_2 \delta\varepsilon_4^{\circ}$$

$$+ R_2\delta\varkappa_4^2 + Q_1\delta\epsilon_5^\circ + R_1\delta\varkappa_5^2 + N_6\delta\epsilon_6^\circ + M_6\delta\varkappa_6^\circ + P_6\delta\varkappa_6^2 \Big\} dxdy \qquad (48)$$

where the stress resultants are defined to be

$$(N_i, M_i, P_i) = \int_{-h/2}^{h/2} \sigma_i (1, z, z^3) dz \qquad (i = 1, 2, 6)$$

$$(Q_2, R_2) = \int_{-h/2}^{h/2} \sigma_4 (1, z^2) dz \qquad (49)$$

$$(Q_1, R_1) = \int_{-h/2}^{h/2} \sigma_5 (1, z^2) dz$$

as given in [10]. N_i, M_i, and Q_i are found in conventional laminated plate theories (see [13]). P_i and R_i are additional terms unique to the higher order theory and may be thought of as moment and shear corrections respectively due to the transverse shear strain varying parabolically through the thickness.

We now substitute Eq.(22) into Eq.(49) to give

$$\delta U = \iint_R \Big\{ N_1\delta u_{o,x} + M_1\delta\psi_{x,x} + P_1\Big[-(4/3h^2)(\delta\psi_{x,x} + \delta w_{,xx}) \Big] + N_2\delta u_{o,y}$$

$$+ M_2\psi_{y,y} + P_2\Big[-(4/3h^2)(\delta\psi_{y,y} + \delta w_{,yy}) \Big] + Q_2(\delta\psi_y + \delta w_{,y})$$

$$+ R_2\Big[-(4/h^2)(\delta\psi_y + \delta w_{,y}) \Big] + Q_1(\delta\psi_x + \delta w_{,x})$$

$$+ R_1\Big[-(4/h^2)(\delta\psi_x + \delta w_{,x}) \Big] + N_6(\delta u_{o,y} + \delta v_{o,x}) + M_6(\delta\psi_{x,y} + \delta\psi_{y,x})$$

$$+ P_6\left[-(4/3h^2)(\delta\psi_{x},_{y} + \delta\psi_{y},_{x} + 2\delta w,_{xy}) \right] \Big\} dxdy \tag{50}$$

We now integrate by parts using the following formulae [15]

$$\iint_R G\ H,_x\ dxdy = -\int_0^b\int_0^a H\ G,_x\ dxdy + \int_0^b G\ H \Big|_0^a dy$$

$$\iint_R G\ H,_y\ dxdy = -\int_0^b\int_0^a H\ G,_y\ dxdy - \int_0^a G\ H \Big|_b^0 dx$$

$$\int_0^a G\ H,_x \Big|_b^0 dx = G\ H \Big|_b^0 \Big|_0^a - \int_0^a H\ G,_x \Big|_b^0 dx$$

$$\int_0^b G\ H,_y \Big|_0^a dy = G\ H \Big|_0^b \Big|_0^a - \int_0^b H\ G,_y \Big|_0^a dy \tag{51}$$

and the plate dimensions of Figure 2.3 to yield

$$\delta U = \int_0^b \int_0^a \Big\{ -\delta u_o N_1,_x - \delta\psi_x M_1,_x + (4/3h^2)\delta\psi_x P_1,_x - (4/3h^2)\delta w P_1,_{xx}$$

$$-\delta v_o N_2,_y - \delta\psi_y M_2,_y + (4/3h^2)\delta\psi_y P_2,_y - (4/3h^2)\delta w P_2,_{yy} + \delta\psi_y Q_2 - \delta w Q_2,_y$$

$$- (4/h^2)\delta\psi_y R_2 + (4/h^2)\delta w R_2,_y + \delta\psi_x Q_1 - \delta w Q_1,_x - (4/h^2)\delta\psi_x R_1$$

$$+ (4/h^2)\delta w R_1,_x - \delta u_o N_6,_y - \delta v_o N_6,_x - \delta\psi_x M_6,_y - \delta\psi_y M_6,_x + (4/3h^2)\delta\psi_x P_6,_y$$

$$+ (4/3h^2)\delta\psi_y P_6,_x - (8/3h^2)\delta w P_6,_{xy} \Big\} dxdy + \int_0^b \Big\{ N_1\delta u_o + M_1\delta\psi_x$$

$$- (4/3h^2)P_1\delta\psi_x + (4/3h^2)P_1,_x\delta w - (4/3h^2)P_1\delta w,_x + Q_1\delta w - (4/h^2)R_1\delta w$$

$$+ N_6\delta v_o + M_6\delta\psi_y - (4/3h^2)P_6\delta\psi_y + (8/3h^2)P_{6,y}\delta w \Big\} \Big|_0^a dy$$

$$+ \int_0^a \Big\{ -N_2\delta v_o - M_2\delta\psi_y + (4/3h^2)P_2\delta\psi_y - (4/3h^2)P_{2,y}\delta w + (4/3h^2)P_2\delta w_{,y}$$

$$- Q_2\delta w + (4/h^2)R_2\delta w - N_6\delta u_o - M_6\delta\psi_x + (4/3h^2)P_6\delta\psi_x$$

$$- (8/3h^2)P_{6,x}\delta w \Big\} \Big|_b^0 dx + (8/3h^2)P_6\delta w \Big|_b^0\Big|_0^a \qquad (52)$$

Looking closely at the above equation, one notices that the boundary integrals each contain a term $\delta w_{,x}$ or $\delta w_{,y}$. These terms appear now where else. We can deal with these terms by considering the fact that the rotation of the midplane of the plate ($w_{,x}$ and $w_{,y}$) is the sum of the rotation due to bending (ψ_x and ψ_y) and the rotation due to shear (γ_{xz} and γ_{yz}). Thus

$$w_{,x} = \psi_x + \gamma_{xz} \quad ; \quad w_{,y} = \psi_y + \gamma_{yz} \qquad (53)$$

For a simply supported boundary , the edges of the plate are free to rotate and edges remain perpendicular to the midplane. They have no shear strain component. The edges of a clamped boundary are restrained from rotation and again the edges stay perpendicular to the midplane and have zero shear strain. We may therefore assume that $w_{,x} = \psi_x$ and $w_{,y} = \psi_y$. Taking the first variation we obtain $\delta w_{,x} = \delta\psi_x$ and $\delta w_{,y} = \delta\psi_y$ which keeps the problem a function of three variables instead of five.

We substitute relations from the above assumption and collect terms to give

$$\delta U = \int_0^b \int_0^a \Bigg\{ \Big[-N_1,_x \; -N_6,_y \Big] \delta u_o + \Big[-N_2,_y \; -N_6,_x \Big] \delta v_o$$

$$+ \Big[-(4/3h^2)P_1,_{xx} - (4/3h^2)P_2,_{yy} - Q_2,_y + (4/h^2)R_2,_y - Q_1,_x + (4/h^2)R_1,_x$$

$$- (8/3h^2)P_6,_{xy} \Big] \delta w + \Big[-M_1,_x + (4/3h^2)P_1,_x + Q_1 - (4/h^2)R_1 - M_6,_y$$

$$+ (4/3h^2)P_6,_y \Big] \delta\psi_x + \Big[-M_2,_y + (4/3h^2)P_2,_y + Q_2 - (4/h^2)R_2 - M_6,_x$$

$$+ (4/3h^2)P_6,_x \Big] \delta\psi_y \Bigg\} dxdy + \int_0^b \Bigg\{ \Big[N_1 \Big] \delta u_o + \Big[N_6 \Big] \delta v_o$$

$$+ \Big[(4/3h^2)P_1,_x + Q_1 - (4/h^2)R_1 + (8/3h^2)P_6,_y \Big] \delta w$$

$$+ \Big[M_1 - (8/3h^2)P_1 \Big] \delta\psi_x + \Big[M_6 - (4/3h^2)P_6 \Big] \delta\psi_y \Bigg\} \Big|_0^a dy$$

$$+ \int_0^a \Bigg\{ \Big[-N_6 \Big] \delta u_o + \Big[-N_2 \Big] \delta v_o$$

$$+ \Big[-(4/3h^2)P_2,_y - Q_2 + (4/h^2)R_2 + (8/3h^2)P_6,_x \Big] \delta w$$

$$+ \Big[-M_6 + (4/3h^2)P_6 \Big] \delta\psi_x + \Big[-M_2 + (8/3h^2)P_2 \Big] \delta\psi_y \Bigg\} \Big|_b^0 dx$$

$$+ (8/3h^2)P_6 \delta w \Big|_b^0 \Big|_0^a \qquad (54)$$

Finally, we substitute the first variation of kinetic energy Eq.(37), of potential energy due to external forces Eq.(42), and of strain energy Eq.(54) into Hamilton's principle Eq.(30) to give

$$\int_{t_1}^{t_2} \int_0^b \int_0^a \Bigg\{ \Big(I_1 \dot{u}_o + \bar{I}_2 \dot{\psi}_x - (4/3h^2) I_4 \dot{w},_x \Big) \delta \dot{u}_o$$

$$+ \left(I_1 \dot{v}_o + \bar{I}_2 \dot{\psi}_y - (4/3h^2) I_4 \dot{w},_y \right) \delta \dot{v}_o$$

$$+ \left((4/3h^2) I_4 \dot{u}_{o,x} + (4/3h^2) \bar{I}_5 \dot{\psi}_{x,x} - (4/3h^2)^2 I_7 \dot{w},_{xx} + (4/3h^2) I_4 \dot{v}_{o,y} \right.$$

$$\left. + (4/3h^2) \bar{I}_5 \dot{\psi}_{y,y} - (4/3h^2)^2 I_7 \dot{w},_{yy} + I_1 \dot{w} \right) \delta \dot{w}$$

$$\left(\bar{I}_2 \dot{u}_o + \bar{I}_3 \dot{\psi}_x - (4/3h^2) \bar{I}_5 \dot{w},_x \right) \delta \dot{\psi}_x$$

$$+ \left(\bar{I}_2 \dot{v}_o + \bar{I}_3 \dot{\psi}_y - (4/3h^2) \bar{I}_5 \dot{w},_y \right) \delta \dot{\psi}_y$$

$$- \left(-\bar{N}_1 w,_{xx} - \bar{N}_2 w,_{yy} - 2 \bar{N}_6 w,_{xy} \right) \delta w$$

$$- \left[\left(-N_1,_x - N_6,_y \right) \delta u_o + \left(-N_2,_y - N_6,_x \right) \delta v_o \right.$$

$$+ \left(-(4/3h^2) P_1,_{xx} - (4/3h^2) P_2,_{yy} - Q_2,_y + (4/h^2) R_2,_y - Q_1,_x + (4/h^2) R_1,_x \right.$$

$$\left. - (8/3h^2) P_6,_{xy} \right) \delta w + \left(-M_1,_x + (4/3h^2) P_1,_x + Q_1 - (4/h^2) R_1 - M_6,_y \right.$$

$$\left. + (4/3h^2) P_6,_y \right) \delta \psi_x + \left(-M_2,_y + (4/3h^2) P_2,_y + Q_2 - (4/h^2) R_2 - M_6,_x \right.$$

$$\left. \left. + (4/3h^2) P_6,_x \right) \delta \psi_y \right] \Bigg\} \, dxdydt$$

$$- \int_{t_1}^{t_2} \int_0^b \Bigg\{ \left(\bar{N}_1 w,_x + \bar{N}_6 w,_y \right) \delta w + \left(N_1 \right) \delta u_o + \left(N_6 \right) \delta v_o$$

$$+ \left((4/3h^2) P_1,_x + Q_1 - (4/h^2) R_1 + (8/3h^2) P_6,_y \right) \delta w$$

$$+ \left(M_1 - (8/3h^2) P_1 \right) \delta \psi_x + \left(M_6 - (4/3h^2) P_6 \right) \delta \psi_y \Bigg\} \Bigg|_0^a dydt$$

$$- \int_{t_1}^{t_2} \int_0^a \Bigg\{ \left(- \bar{N}_2 w,_y - \bar{N}_6 w,_x \right) \delta w + \left(-N_6 \right) \delta u_o + \left(-N_2 \right) \delta v_o$$

$$+ \left(-(4/3h^2) P_2,_y - Q_2 + (4/h^2) R_2 + (8/3h^2) P_6,_x \right) \delta w$$

$$+ \left(-M_6 + (4/3h^2)P_6 \right) \delta\psi_x + \left(-M_2 + (8/3h^2)P_2 \right) \delta\psi_y \Big\} \Big|_b^0 dxdt$$

$$- \int_{t_1}^{t_2} (8/3h^2)P_6 \delta w \Big|_b^0 \Big|_0^a dt = 0 \tag{55}$$

We now integrate by parts with respect to time and use $\delta w = \delta\psi_x = \delta\psi_y = 0$ in the evaluation term at time $t = t_1$ and $t = t_2$. This results from the virtual displacements being zero at time limits t_1 and t_2. The resulting equation is

$$\int_{t_1}^{t_2}\int_0^b \int_0^a \Big\{ \left(- I_1\ddot{u}_o - \bar{I}_2\ddot{\psi}_x + (4/3h^2)I_4\ddot{w},_x + N_1,_x + N_6,_y \right) \delta u_o$$

$$+ \left(-I_1\ddot{v}_o - \bar{I}_2\ddot{\psi}_y + (4/3h^2)I_4\ddot{w},_y + N_2,_y + N_6,_x \right) \delta v_o$$

$$+ \Big(- I_1\ddot{w} + (4/3h^2)^2 I_7(\ddot{w},_{xx} + \ddot{w},_{yy}) - (4/3h^2)I_4(\ddot{u}_o,_x + \ddot{v}_o,_y)$$

$$- (4/3h^2)\bar{I}_5(\ddot{\psi}_x,_x + \ddot{\psi}_y,_y) + Q_1,_x + Q_2,_y + \bar{N}_1 w,_{xx} + \bar{N}_2 w,_{yy} -2 \bar{N}_6 w,_{xy}$$

$$+ (4/h^2) \left(R_2,_y + R_1,_x \right) + (4/3h^2)\left(P_1,_{xx} + P_2,_{yy} + 2 P_6,_{xy} \right) \Big) \delta w$$

$$+ \Big(- \bar{I}_2\ddot{u}_o - \bar{I}_3\ddot{\psi}_x + (4/3h^2)\bar{I}_5\ddot{w},_x + M_1,_x + M_6,_y - Q_1 + (4/h^2)R_1$$

$$- (4/3h^2)(P_1,_x + P_6,_y) \Big) \delta\psi_x + \Big(- \bar{I}_2\ddot{v}_o - \bar{I}_3\ddot{\psi}_y + (4/3h^2)\bar{I}_5\ddot{w},_y$$

$$+ M_2,_y + M_6,_x - Q_2 + (4/h^2)R_2 - (4/3h^2)(P_2,_y + P_6,_x) \Big) \delta\psi_y \Big\} dxdydt$$

$$- \int_{t_1}^{t_2}\int_0^b \Big\{ N_1\delta u_o + N_6\delta v_o + \Big(\bar{N}_1 w,_x + \bar{N}_6 w,_y + Q_1 - (4/h^2)R_1$$

$$+ (4/3h^2)\left(P_1,_x + 2 P_6,_y \right) \Big) \delta w + \left(M_1 - (8/3h^2)P_1 \right) \delta\psi_x$$

$$+ \left(M_6 - (4/3h^2)P_6 \right) \delta\psi_y \Bigg\} \Bigg|_0^a dydt$$

$$- \int_{t_1}^{t_2}\int_0^a \Bigg\{ -N_6\delta u_o - N_2\delta v_o + \Big(-\bar{N}_2 w,_y - \bar{N}_6 w,_x - Q_2 + (4/h^2)R_2$$

$$- (4/3h^2)(P_2,_y + 2\, P_6,_x) \Big) \delta w + \Big(-M_6 - (4/3h^2)P_6 \Big) \delta\psi_x$$

$$+ \Big(-M_2 + (8/3h^2)P_2 \Big) \delta\psi_y \Bigg\} \Bigg|_b^0 dxdt - \int_{t_1}^{t_2} (8/3h^2)P_6\delta w \Bigg|_b^0 \Bigg|_0^a dt = 0 \quad (56)$$

Since the variation cannot be zero over the region, the coefficients must go to zero and the equations of motion are

$$\delta u_o: \qquad N_1,_x + N_6,_y = I_1\ddot{u}_o + \bar{I}_2\ddot{\psi}_x \underline{- (4/3h^2)I_4\ddot{w},_x}$$

$$\delta v_o: \qquad N_2,_y + N_6,_x = I_1\ddot{v}_o + \bar{I}_2\ddot{\psi}_y \underline{- (4/3h^2)I_4\ddot{w},_y}$$

$$\delta w: \qquad Q_1,_x + Q_2,_y + \bar{N}_1 w,_{xx} + \bar{N}_2 w,_{yy} - 2\,\bar{N}_6 w,_{xy} \underline{- (4/h^2)\,(R_2,_y + R_1,_x)}$$

$$\underline{+ (4/3h^2)(P_1,_{xx} + P_2,_{yy} + 2\, P_6,_{xy})} = I_1\ddot{w} \underline{- (4/3h^2)^2 I_7(\ddot{w},_{xx} + \ddot{w},_{yy})}$$

$$\underline{+ (4/3h^2)I_4(\ddot{u}_o,_x + \ddot{v}_o,_y) + (4/3h^2)\bar{I}_5(\ddot{\psi}_x,_x + \ddot{\psi}_y,_y)} \qquad (57)$$

$$\delta\psi_x : M_1,_x + M_6,_y - Q_1 \underline{+ (4/h^2)R_1 - (4/3h^2)(P_1,_x + P_6,_y)}$$

$$= \bar{I}_2\ddot{u}_o + \bar{I}_3\ddot{\psi}_x \underline{- (4/3h^2)\bar{I}_5\ddot{w},_x}$$

$$\delta\psi_y : M_2,_y + M_6,_x - Q_2 \underline{+ (4/h^2)R_2 - (4/3h^2)(P_2,_y + P_6,_x)}$$

$$= \bar{I}_2\ddot{v}_o + \bar{I}_3\ddot{\psi}_y \underline{- (4/3h^2)\bar{I}_5\ddot{w},_y}$$

which are identical to those in [12]. The underlined terms are ones not seen in the Mindlin-type plate theories. In addition to solution of the equations of motion, the following boundary conditions must be satisfied (each term must vanish since the variations are arbitrary)

$$\delta u_o : \left(\int_0^a N_6 \Big|_b^0 dx - \int_0^b N_1 \Big|_0^a dy \right) \delta u_o = 0$$

$$\delta v_o : \left(\int_0^a N_2 \Big|_b^0 dx - \int_0^b N_6 \Big|_0^a dy \right) \delta v_o = 0$$

δw :

$$\left(\int_0^a \left\{ \bar{N}_2 w,_y + \bar{N}_6 w,_x + Q_2 - (4/h^2)R_2 + (4/3h^2)\left(P_2,_y + 2P_6,_x\right) \right\} \Big|_b^0 dx \right.$$

$$+ \int_0^b \left\{ \bar{N}_1 w,_x + \bar{N}_6 w,_y + Q_1 - (4/h^2)R_1 + (4/3h^2)\left(P_1,_x + 2P_6,_y\right) \right\} \Big|_0^a dy$$

$$\left. - (8/3h^2)P_6 \Big|_b^0 \Big|_0^a \right) \delta w = 0 \tag{58}$$

$$\delta\psi_x : \left(\int_0^a \left(M_6 - (4/3h^2)P_6\right)\Big|_b^0 dx + \int_0^b \left(-M_1 + (8/3h^2)P_1\right)\Big|_0^a dy \right) \delta\psi_x = 0$$

$$\delta\psi_y : \left(\int_0^a \left(M_2 - (8/3h^2)P_2\right)\Big|_b^0 dx + \int_0^b \left(-M_6 + (4/3h^2)P_6\right)\Big|_0^a dy \right) \delta\psi_y = 0$$

Since we only desire solutions for natural frequencies and buckling loads, only the out-of-plane equations are needed. In other words, the δu_o and δv_o equations of Eq.(57) and (58) can be dropped. Note that these equations must be included for considering shells.

In order to achieve solutions using the higher order theory, we will approximate a formal solution by combining an equation of motion with its corresponding boundary condition requirement. This will give three equations in terms of three variables.

This work will deal only with symmetric laminates. From Eq.(33) this implies that $I_2 = I_4 = \bar{I}_2 = 0$ and gives

δw :

$$\int_{t_1}^{t_2}\int_0^b \int_0^a \Big\{ Q_{1,x} + Q_{2,y} + \bar{N}_1 w_{,xx} + \bar{N}_2 w_{,yy} - 2\bar{N}_6 w_{,xy} - (4/h^2)\Big(R_{2,y} + R_{1,x}\Big)$$
$$+ (4/3h^2)\Big(P_{1,xx} + P_{2,yy} + 2P_{6,xy}\Big) - I_1\ddot{w} + (4/3h^2)^2 I_7\Big(\ddot{w}_{,xx} + \ddot{w}_{,yy}\Big)$$
$$- (4/3h^2)\bar{I}_5\Big(\ddot{\psi}_{x,x} + \ddot{\psi}_{y,y}\Big)\Big\} \delta w \, dxdydt$$
$$- \int_{t_1}^{t_2}\int_0^b \Big\{\bar{N}_1 w_{,x} + \bar{N}_6 w_{,y} + Q_1 - (4/h^2)R_1 + (4/3h^2)\Big(P_{1,x} + 2P_{6,y}\Big)\Big\}\Big|_0^a \delta w dydt$$
$$- \int_{t_1}^{t_2}\int_0^a \Big\{-\bar{N}_2 w_{,y} - \bar{N}_6 w_{,x} - Q_2 + (4/h^2)R_2 - (4/3h^2)\Big(P_{2,y} + 2P_{6,x}\Big)\Big\}\Big|_b^0 \delta w dxdt$$
$$- \int_{t_1}^{t_2} (8/3h^2)P_6 \Big|_b^0 \Big|_0^a \delta w dt = 0 \qquad (59)$$

$\delta\psi_x$:

$$\int_{t_1}^{t_2}\int_0^b \int_0^a \Big\{ M_{1,x} + M_{6,y} - Q_1 + (4/h^2)R_1 - (4/3h^2)\Big(P_{1,x} + P_{6,y}\Big)$$
$$- \bar{I}_3\ddot{\psi}_x + (4/3h^2)\bar{I}_5\ddot{w}_{,x}\Big\} \delta\psi_x dxdydt - \int_{t_1}^{t_2}\int_0^b \Big\{M_1 - (8/3h^2)P_1\Big\}\Big|_0^a \delta\psi_x dydt$$

$$- \int_{t_1}^{t_2}\int_0^a \left\{-M_6 + (4/3h^2)P_6\right\} \Big|_b^0 \delta\psi_x dxdt = 0 \qquad (60)$$

$\delta\psi_y$:

$$\int_{t_1}^{t_2}\int_0^b \int_0^a \left\{ M_2,_y + M_6,_x - Q_2 + (4/h^2)R_2 - (4/3h^2)\left(P_2,_y + P_6,_x\right)\right.$$

$$\left. - \bar{I}_3\ddot{\psi}_y + (4/3h^2)\bar{I}_5\ddot{w},_y \right\} \delta\psi_y dxdydt - \int_{t_1}^{t_2}\int_0^b \left\{M_6 - (4/3h^2)P_6\right\} \Big|_0^a \delta\psi_y dydt$$

$$- \int_{t_1}^{t_2}\int_0^a \left\{-M_2 + (8/3h^2)P_2\right\} \Big|_b^0 \delta\psi_y dxdt = 0 \qquad (61)$$

from Eq.(56).

If we assume the time dependance to be harmonic, the problem is independant of time and Eqs.(59), (60), and (61) can be written as

δw :

$$\int_0^b \int_0^a \left\{Q_1,_x + Q_2,_y + \bar{N}_1 w,_{xx} + \bar{N}_2 w,_{yy} -2\bar{N}_6 w,_{xy} - (4/h^2)\left(R_2,_y + R_1,_x\right)\right.$$

$$+ (4/3h^2)\left(P_1,_{xx} + P_2,_{yy} + 2P_6,_{xy}\right) + \omega^2 I_1 w - \omega^2(4/3h^2)^2 I_7\left(w,_{xx} + w,_{yy}\right)$$

$$\left. + \omega^2(4/3h^2)\bar{I}_5\left(\psi_x,_x + \psi_y,_y\right) \right\} \delta w \, dxdy$$

$$- \int_0^b \left\{ N_1 w,_x + \bar{N}_6 w,_y + Q_1 - (4/h^2)R_1 + (4/3h^2)\left(P_1,_x + 2P_6,_y\right) \right\} \Big|_0^a \delta w dy$$

$$- \int_0^a \left\{ -\bar{N}_2 w,_y - \bar{N}_6 w,_x - Q_2 + (4/h^2)R_2 - (4/3h^2)\left(P_2,_y + 2P_6,_x\right) \right\} \Big|_b^0 \delta w dx$$

$$- (8/3h^2)P_6 \Big|_b^0 \Big|_0^a \; \delta w = 0 \tag{62}$$

$\delta\psi_x$:

$$\int_0^b \int_0^a \Big\{ M_1,_x + M_6,_y - Q_1 + (4/h^2)R_1 - (4/3h^2)\Big(P_1,_x + P_6,_y\Big)$$

$$+ \omega^2 \bar{I}_3 \psi_x - \omega^2 (4/3h^2)\bar{I}_5 \ddot{w},_x \Big\} \delta\psi_x dxdy - \int_0^b \Big\{ M_1 - (8/3h^2)\; P_1 \Big\} \Big|_0^a \delta\psi_x dy$$

$$- \int_0^a \Big\{ -M_6 + (4/3h^2)\; P_6 \Big\} \Big|_b^0 \delta\psi_x dx = 0 \tag{63}$$

$\delta\psi_y$:

$$\int_0^b \int_0^a \Big\{ M_2,_y + M_6,_x - Q_2 + (4/h^2)R_2 - (4/3h^2)\Big(P_2,_y + P_6,_x\Big)$$

$$+ \omega^2 \bar{I}_3 \psi_y - \omega^2 (4/3h^2)\bar{I}_5 w,_y \Big\} \delta\psi_y dxdy - \int_0^b \Big\{ M_6 - (4/3h^2)\; P_6 \Big\} \Big|_0^a \delta\psi_y dy$$

$$- \int_0^a \Big\{ -M_2 + (8/3h^2)\; P_2 \Big\} \Big|_b^0 \delta\psi_y dx = 0 \tag{64}$$

We now expand the boundary terms.

δw :

$$\int_0^b \int_0^a \Big\{ Q_1,_x + Q_2,_y + \bar{N}_1 w,_{xx} + \bar{N}_2 w,_{yy} - 2\bar{N}_6 w,_{xy} - (4/h^2)\Big(R_2,_y + R_1,_x\Big)$$

$$+ (4/3h^2)\Big(P_1,_{xx} + P_2,_{yy} + 2P_6,_{xy}\Big) + \omega^2 I_1 w - \omega^2 (4/3h^2)^2 I_7 \Big(w,_{xx} + w,_{yy}\Big)$$

$$+ \omega^2 (4/3h^2)\bar{I}_5 \Big(\psi_x,_x + \psi_y,_y\Big) \Big\} \delta w \; dxdy$$

$$+ \int_0^b \Big\{ \bar{N}_1 w,_x(0,y) + \bar{N}_6 w,_y(0,y) + Q_1(0,y) - (4/h^2)R_1(0,y)$$

$$+ (4/3h^2)\Big(P_1,_x(0,y) + 2P_6,_y(0,y) \Big) - \bar{N}_1 w,_x(a,y) - \bar{N}_6 w,_y(a,y)$$

$$- Q_1(a,y) + (4/h^2)R_1(a,y) - (4/3h^2)\Big(P_1,_x(a,y) + 2P_6,_y(a,y) \Big) \Big\} \delta w dy$$

$$+ \int_0^a \Big\{ \bar{N}_2 w,_y(x,0) + \bar{N}_6 w,_x(x,0) + Q_2(x,0) - (4/h^2)R_2(x,0)$$

$$- (4/3h^2)\Big(P_2,_y(x,0) + 2\ P_6,_x(x,0) \Big) - \bar{N}_2 w,_y(x,b) - \bar{N}_6 w,_x(x,b)$$

$$- Q_2(x,b) + (4/h^2)R_2(x,b) + (4/3h^2)\Big(P_2,_y(x,b) + 2\ P_6,_x(x,b) \Big) \Big\} \delta w dx$$

$$- (8/3h^2)\ \Big(P_6(a,0) - P_6(a,b) - P_6(0,0) - P_6(0,b) \Big)\ \delta w = 0 \qquad (65)$$

$\delta\psi_x$:

$$\int_0^b \int_0^a \Big\{ M_1,_x + M_6,_y - Q_1 + (4/h^2)R_1 - (4/3h^2)\Big(P_1,_x + P_6,_y \Big)$$

$$+ \omega^2 \bar{I}_3 \psi_x - \omega^2 (4/3h^2) \bar{I}_5 \ddot{w},_x \Big\} \delta\psi_x dxdy$$

$$+ \int_0^b \Big\{ M_1(0,y) - (8/3h^2)P_1(0,y) - M_1(a,y) + (8/3h^2)P_1(a,y) \Big\} \delta\psi_x dy$$

$$+ \int_0^a \Big\{ M_6(x,0) - (4/3h^2)P_6(x,0) - M_6(x,b) + (4/3h^2)P_6(x,b) \Big\} \delta\psi_x dx = 0 \qquad (66)$$

$\delta\psi_y$:

$$\int_0^b \int_0^a \Big\{ M_2,_y + M_6,_x - Q_2 + (4/h^2)R_2 - (4/3h^2)\Big(P_2,_y + P_6,_x \Big)$$

$$+ \omega^2 \bar{I}_3 \psi_y - \omega^2 (4/3h^2) \bar{I}_5 w_{,y} \Bigg\} \delta\psi_y dxdy$$

$$+ \int_0^b \Bigg\{ M_6(0,y) - (4/3h^2) P_6(0,y) - M_6(a,y) + (4/3h^2) P_6(a,y) \Bigg\} \delta\psi_y dy$$

$$+ \int_0^a \Bigg\{ M_2(x,0) - (8/3h^2) P_2(x,0) - M_2(x,b) + (8/3h^2) P_2(x,b) \Bigg\} \delta\psi_y dx = 0 \quad (67)$$

In order to relate the stress resultants to the total strains we use from [10]

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} N_1 \\ N_2 \\ N_6 \end{array} \right\} \\ \left\{ \begin{array}{c} M_1 \\ M_2 \\ M_6 \end{array} \right\} \\ \left\{ \begin{array}{c} P_1 \\ P_2 \\ P_6 \end{array} \right\} \end{array} \right\} = \left[\begin{array}{ccc} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \\ & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} & \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \\ \text{symmetric} & & \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \end{array} \right] \left\{ \begin{array}{c} \left\{ \begin{array}{c} \varepsilon_1^o \\ \varepsilon_2^o \\ \varepsilon_6^o \end{array} \right\} \\ \left\{ \begin{array}{c} \kappa_1^o \\ \kappa_2^o \\ \kappa_6^o \end{array} \right\} \\ \left\{ \begin{array}{c} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{array} \right\} \end{array} \right\} \quad (68)$$

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} Q_2 \\ Q_1 \end{array} \right\} \\ \left\{ \begin{array}{c} R_2 \\ R_1 \end{array} \right\} \end{array} \right\} = \left[\begin{array}{cc} \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} & \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \\ \text{sym.} & \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \end{array} \right] \left\{ \begin{array}{c} \left\{ \begin{array}{c} \varepsilon_4^o \\ \varepsilon_5^o \end{array} \right\} \\ \left\{ \begin{array}{c} \kappa_4^2 \\ \kappa_5^2 \end{array} \right\} \end{array} \right\}$$

where the plate stiffnesses are defined to be from [10]

$$A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij} \, (1, z, z^2, z^3, z^4, z^6) \, dz \qquad (i,j=1,2,6)$$

$$\tag{69}$$

$$A_{ij}, D_{ij}, F_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij} \, (1, z^2, z^4) \, dz \qquad (i,j=4,5)$$

and the $\bar{Q}_{ij}$'s are the transformed reduced stiffnesses.

For symmetric laminates $B_{ij} = E_{ij} = 0$ (from Eq.(69), and Eq.(68) becomes

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^o \\ \varepsilon_2^o \\ \varepsilon_6^o \end{Bmatrix}$$

$$\begin{Bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} \\ \begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} & \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \\ \text{sym.} & \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \kappa_1^o \\ \kappa_2^o \\ \kappa_6^o \end{Bmatrix} \\ \begin{Bmatrix} \kappa_1^2 \\ \kappa_2^2 \\ \kappa_6^2 \end{Bmatrix} \end{Bmatrix} \tag{70}$$

$$\left\{\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix} \atop \begin{Bmatrix} R_2 \\ R_1 \end{Bmatrix}\right\} = \begin{bmatrix} \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} & \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \\ \text{sym.} & \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \end{bmatrix} \left\{\begin{Bmatrix} \varepsilon_4^o \\ \varepsilon_5^o \end{Bmatrix} \atop \begin{Bmatrix} \kappa_4^2 \\ \kappa_5^2 \end{Bmatrix}\right\}$$

The problem at hand is independant of the extensional stress resultants N_1, N_2, and N_6 and the first relation of Eq.(70) will not be needed.

Using the strain relations of Eq.(22) we obtain

$$M_1 = D_{11}\psi_{x},_x + D_{12}\psi_{y},_y + D_{16}(\psi_{x},_y+\psi_{y},_x) - (4/3h^2)F_{11}(\psi_{x},_x+w,_{xx})$$
$$- (4/3h^2)F_{12}(\psi_{y},_y+w,_{yy}) - (4/3h^2)F_{16}(\psi_{x},_y+\psi_{y},_x+2w,_{xy})$$

$$M_2 = D_{12}\psi_{x},_x + D_{22}\psi_{y},_y + D_{26}(\psi_{x},_y+\psi_{y},_x) - (4/3h^2)F_{12}(\psi_{x},_x+w,_{xx})$$
$$- (4/3h^2)F_{22}(\psi_{y},_y+w,_{yy}) - (4/3h^2)F_{26}(\psi_{x},_y+\psi_{y},_x+2w,_{xy})$$

$$M_6 = D_{16}\psi_{x},_x + D_{26}\psi_{y},_y + D_{66}(\psi_{x},_y+\psi_{y},_x) - (4/3h^2)F_{16}(\psi_{x},_x+w,_{xx})$$
$$- (4/3h^2)F_{26}(\psi_{y},_y+w,_{yy}) - (4/3h^2)F_{66}(\psi_{x},_y+\psi_{y},_x+2w,_{xy})$$

$$P_1 = F_{11}\psi_{x},_x + F_{12}\psi_{y},_y + F_{16}(\psi_{x},_y+\psi_{y},_x) - (4/3h^2)H_{11}(\psi_{x},_x+w,_{xx})$$
$$- (4/3h^2)H_{12}(\psi_{y},_y+w,_{yy}) - (4/3h^2)H_{16}(\psi_{x},_y+\psi_{y},_x+2w,_{xy})$$

$$P_2 = F_{12}\psi_{x},_x + F_{22}\psi_{y},_y + F_{26}(\psi_{x},_y+\psi_{y},_x) - (4/3h^2)H_{12}(\psi_{x},_x+w,_{xx})$$
$$- (4/3h^2)H_{22}(\psi_{y},_y+w,_{yy}) - (4/3h^2)H_{26}(\psi_{x},_y+\psi_{y},_x+2w,_{xy}) \qquad (71)$$

$$P_6 = F_{16}\psi_{x,x} + F_{26}\psi_{y,y} + F_{66}(\psi_{x,y}+\psi_{y,x}) - (4/3h^2)H_{16}(\psi_{x,x}+w_{,xx})$$
$$- (4/3h^2)H_{26}(\psi_{y,y}+w_{,yy}) - (4/3h^2)H_{66}(\psi_{x,y}+\psi_{y,x}+2w_{,xy})$$

$$Q_2 = (\ A_{44}-(4/h^2)D_{44}\ )(\psi_y+w_{,y}) + (\ A_{45}-(4/h^2)D_{45}\ )(\psi_x+w_{,x})$$

$$Q_1 = (\ A_{45}-(4/h^2)D_{45}\ )(\psi_y+w_{,y}) + (\ A_{55}-(4/h^2)D_{55}\ )(\psi_x+w_{,x})$$

$$R_2 = (\ D_{44}-(4/h^2)F_{44}\ )(\psi_y+w_{,y}) + (\ D_{45}-(4/h^2)F_{45}\ )(\psi_x+w_{,x})$$

$$R_1 = (\ D_{45}-(4/h^2)F_{45}\ )(\psi_y+w_{,y}) + (\ D_{55}-(4/h^2)F_{55}\ )(\psi_x+w_{,x})$$

These relations are now substituted into Eqs.(65), (66), and (67). We now have three equations of motion written in terms of the three dependant variables ψ_x, ψ_y, and w (the bending rotations and normal displacement).

δw :

$$\int_0^b\int_0^a \Big\{ (A_{45}-(4/h^2)D_{45})(\psi_{y},_{x}+w,_{xy}) + (A_{55}-(4/h^2)D_{55})(\psi_{x},_{x}+w,_{xx})$$

$$+ (A_{44}-(4/h^2)D_{44})(\psi_{y},_{y}+w,_{yy}) + (A_{45}-(4/h^2)D_{45})(\psi_{x},_{y}+w,_{xy})$$

$$+ \bar{N}_1 w,_{xx} + \bar{N}_2 w,_{yy} + \bar{N}_6 w,_{xy} - (4/h^2)(D_{45}-(4/h^2)F_{45})(\psi_{y},_{x}+w,_{xy})$$

$$- (4/h^2)(D_{55}-(4/h^2)F_{55})(\psi_{x},_{x}+w,_{xx}) - (4/h^2)(D_{44}-(4/h^2)F_{44})$$

$$(\psi_{y},_{y}+w,_{yy}) - (4/h^2)(D_{45}-(4/h^2)F_{45})(\psi_{x},_{y}+w,_{xy}) + (4/3h^2)F_{11}\psi_{x},_{xxx}$$

$$+ (4/3h^2)F_{12}\psi_{y},_{xxy} + (4/3h^2)F_{16}(\psi_{x},_{xxy}+\psi_{y},_{xxx})$$

$$- (16/9h^4)H_{11}(\psi_{x},_{xxx}+w,_{xxxx}) - (16/9h^4)H_{12}(\psi_{y},_{xxy}+w,_{xxyy})$$

$$- (16/9h^4)H_{16}(\psi_{x},_{xxy}+\psi_{y},_{xxx}+2w,_{xxxy}) + (4/3h^2)F_{12}\psi_{x},_{xyy}$$

$$+ (4/3h^2)F_{22}\psi_{y},_{yyy} + (4/3h^2)F_{26}(\psi_{x},_{yyy}+\psi_{y},_{xyy})$$

$$- (16/9h^4)H_{12}(\psi_{x},_{xyy}+w,_{xxyy}) - (16/9h^4)H_{22}(\psi_{y},_{yyy}+w,_{yyyy})$$

$$- (16/9h^4)H_{26}(\psi_{x},_{yyy}+\psi_{y},_{xyy}+2w,_{xyyy}) + (8/3h^2)F_{16}\psi_{x},_{xxy}$$

$$+ (8/3h^2)F_{26}\psi_{y},_{xyy} + (8/3h^2)F_{66}(\psi_{x},_{xyy}+\psi_{y},_{xxy})$$

$$- (32/9h^4)H_{16}(\psi_{x},_{xxy}+w,_{xxxy}) - (32/9h^4)H_{26}(\psi_{y},_{xyy}+w,_{xyyy})$$

$$- (32/9h^4)H_{66}(\psi_{x},_{xxy}+\psi_{y},_{xxy}+2w,_{xxyy}) + \omega^2 I_1 w - \omega^2(4/3h^2)^2 I_7(w,_{xx} + w,_{yy})$$

$$+ \omega^2(4/3h^2)\bar{I}_5(\psi_{x},_{x} + \psi_{y},_{y}) \Big\} \delta w \, dxdy$$

$$+ \int_0^b \Big\{ \bar{N}_1 w,_{x}(0,y) + \bar{N}_6 w,_{y}(0,y) + (A_{45}-(4/h^2)D_{45})(\psi_{y}(0,y)+w,_{y}(0,y))$$

$$+ (A_{55}-(4/h^2)D_{55})(\Psi_x(0,y)+w,_x(0,y)) - (4/h^2)(D_{45}-(4/h^2)F_{45})$$

$$(\Psi_y(0,y)+w,_y(0,y)) - (4/h^2)(D_{55}-(4/h^2)F_{55})(\Psi_x(0,y)+w,_x(0,y))$$

$$+ (4/3h^2)F_{11}\Psi_x,_{xx}(0,y) + (4/3h^2)F_{12}\Psi_y,_{xy}(0,y) + (4/3h^2)F_{16}$$

$$(\Psi_x,_{xy}(0,y)+\Psi_y,_{xx}(0,y)) - (16/9h^4)H_{11}(\Psi_x,_{xx}(0,y)+w,_{xxx}(0,y))$$

$$- (16/9h^4)H_{12}(\Psi_y,_{xy}(0,y)+w,_{xyy}(0,y)) - (16/9h^4)H_{16}$$

$$(\Psi_x,_{xy}(0,y)+\Psi_y,_{xx}(0,y)+2w,_{xxy}(0,y)) + (8/3h^2)F_{16}\Psi_x,_{xy}(0,y)$$

$$+ (8/3h^2)F_{26}\Psi_y,_{yy}(0,y) + (8/3h^2)F_{66}(\Psi_x,_{yy}(0,y)+\Psi_y,_{xy}(0,y))$$

$$- (32/9h^4)H_{16}(\Psi_x,_{xy}(0,y)+w,_{xxy}(0,y)) - (32/9h^4)H_{26}(\Psi_y,_{yy}(0,y)+w,_{yyy}(0,y))$$

$$- (32/9h^4)H_{66}(\Psi_x,_{yy}(0,y)+\Psi_y,_{xy}(0,y)+2w,_{xyy}(0,y))$$

$$-\bar{N}_1w,_x(a,y) - \bar{N}_6w,_y(a,y) - (A_{45}-(4/h^2)D_{45})(\Psi_y(a,y)+w,_y(a,y))$$

$$- (A_{55}-(4/h^2)D_{55})(\Psi_x(a,y)+w,_x(a,y)) + (4/h^2)(D_{45}-(4/h^2)F_{45})$$

$$(\Psi_y(a,y)+w,_y(a,y)) + (4/h^2)(D_{55}-(4/h^2)F_{55})(\Psi_x(a,y)+w,_x(a,y))$$

$$- (4/3h^2)F_{11}\Psi_x,_{xx}(a,y) - (4/3h^2)F_{12}\Psi_y,_{xy}(a,y) - (4/3h^2)F_{16}$$

$$(\Psi_x,_{xy}(a,y)+\Psi_y,_{xx}(a,y)) + (16/9h^4)H_{11}(\Psi_x,_{xx}(a,y)+w,_{xxx}(a,y))$$

$$+ (16/9h^4)H_{12}(\Psi_y,_{xy}(a,y)+w,_{xyy}(a,y)) + (16/9h^4)H_{16}$$

$$(\Psi_x,_{xy}(a,y)+\Psi_y,_{xx}(a,y)+2w,_{xxy}(a,y)) - (8/3h^2)F_{16}\Psi_x,_{xy}(a,y)$$

$$- (8/3h^2)F_{26}\Psi_y,_{yy}(a,y) - (8/3h^2)F_{66}(\Psi_x,_{yy}(a,y)+\Psi_y,_{xy}(a,y))$$

$$+ (32/9h^4)H_{16}(\psi_x,_{xy}(a,y)+w,_{xxy}(a,y)) + (32/9h^4)H_{26}(\psi_y,_{yy}(a,y)+w,_{yyy}(a,y))$$

$$\left. + (32/9h^4)H_{66}(\psi_x,_{yy}(a,y)+\psi_y,_{xy}(a,y)+2w,_{xyy}(a,y)) \right\} \delta w \; dy$$

$$+ \int_0^a \left\{ \bar{N}_2 w,_y(x,0) + \bar{N}_6 w,_x(x,0) + (A_{44}-(4/h^2)D_{44})(\psi_y(x,0)+w,_y(x,0)) \right.$$

$$+ (A_{45}-(4/h^2)D_{45})(\psi_x(x,0)+w,_x(x,0)) - (4/h^2)(D_{44}-(4/h^2)F_{44})$$

$$(\psi_y(x,0)+w,_y(x,0)) - (4/h^2)(D_{45}-(4/h^2)F_{45})(\psi_x(x,0)+w,_x(x,0))$$

$$- (4/3h^2)F_{12}\psi_x,_{xy}(x,0) - (4/3h^2)F_{22}\psi_y,_{yy}(x,0) - (4/3h^2)F_{26}$$

$$(\psi_x,_{yy}(x,0)+\psi_y,_{xy}(x,0)) + (16/9h^4)H_{12}(\psi_x,_{xy}(x,0)+w,_{xxy}(x,0))$$

$$+ (16/9h^4)H_{22}(\psi_y,_{yy}(x,0)+w,_{yyy}(x,0)) + (16/9h^4)H_{26}$$

$$(\psi_x,_{yy}(x,0)+\psi_y,_{xy}(x,0)+2w,_{xyy}(x,0)) - (8/3h^2)F_{16}\psi_x,_{xx}(x,0)$$

$$- (8/3h^2)F_{26}\psi_y,_{xy}(x,0) - (8/3h^2)F_{66}(\psi_x,_{xy}(x,0)+\psi_y,_{xx}(x,0))$$

$$+ (32/9h^4)H_{16}(\psi_x,_{xx}(x,0)+w,_{xxx}(x,0)) + (32/9h^4)H_{26}$$

$$(\psi_y,_{xy}(x,0)+w,_{xyy}(x,0)) + (32/9h^4)H_{66}(\psi_x,_{xy}(x,0)+\psi_y,_{xx}(x,0)+2w,_{xxy}(x,0))$$

$$-\bar{N}_2 w,_y(x,b) - \bar{N}_6 w,_x(x,b) - (A_{44}-(4/h^2)D_{44})(\psi_y(x,b)+w,_y(x,b))$$

$$- (A_{45}-(4/h^2)D_{45})(\psi_x(x,b)+w,_x(x,b)) + (4/h^2)(D_{44}-(4/h^2)F_{44})$$

$$(\psi_y(x,b)+w,_y(x,b)) + (4/h^2)(D_{45}-(4/h^2)F_{45})(\psi_x(x,b)+w,_x(x,b))$$

$$+ (4/3h^2)F_{12}\psi_x,_{xy}(x,b) + (4/3h^2)F_{22}\psi_y,_{yy}(x,b) + (4/3h^2)F_{26}$$

$$(\psi_x,_{yy}(x,b)+\psi_y,_{xy}(x,b)) - (16/9h^4)H_{12}(\psi_x,_{xy}(x,b)+w,_{xxy}(x,b))$$

$$- (16/9h^4)H_{22}(\psi_{y},_{yy}(x,b)+w,_{yyy}(x,b)) - (16/9h^4)H_{26}$$

$$(\psi_{x},_{yy}(x,b)+\psi_{y},_{xy}(x,b)+2w,_{xyy}(x,b)) + (8/3h^2)F_{16}\psi_{x},_{xx}(x,b)$$

$$+ (8/3h^2)F_{26}\psi_{y},_{xy}(x,b) + (8/3h^2)F_{66}(\psi_{x},_{xy}(x,b)+\psi_{y},_{xx}(x,b))$$

$$- (32/9h^4)H_{16}(\psi_{x},_{xx}(x,b)+w,_{xxx}(x,b)) - (32/9h^4)H_{26}$$

$$(\psi_{y},_{xy}(x,b)+w,_{xyy}(x,b)) - (32/9h^4)H_{66}$$

$$(\psi_{x},_{xy}(x,b)+\psi_{y},_{xx}(x,b)+2w,_{xxy}(x,b)) \Big\} \delta w \, dx$$

$$- (8/3h^2) \Big\{ F_{16}\psi_{x},_{x}(a,0) + F_{26}\psi_{y},_{y}(a,0) + F_{66}(\psi_{x},_{y}(a,0)+\psi_{y},_{x}(a,0))$$

$$- (4/3h^2)H_{16}(\psi_{x},_{x}(a,0)+w,_{xx}(a,0)) - (4/3h^2)H_{26}(\psi_{y},_{y}(a,0)+w,_{yy}(a,0))$$

$$- (4/3h^2)H_{66}(\psi_{x},_{y}(a,0)+\psi_{y},_{x}(a,0)+2w,_{xy}(a,0))$$

$$- F_{16}\psi_{x},_{x}(a,b) - F_{26}\psi_{y},_{y}(a,b) - F_{66}(\psi_{x},_{y}(a,b)+\psi_{y},_{x}(a,b))$$

$$+ (4/3h^2)H_{16}(\psi_{x},_{x}(a,b)+w,_{xx}(a,b)) + (4/3h^2)H_{26}(\psi_{y},_{y}(a,b)+w,_{yy}(a,b))$$

$$+ (4/3h^2)H_{66}(\psi_{x},_{y}(a,b)+\psi_{y},_{x}(a,b)+2w,_{xy}(a,b))$$

$$- F_{16}\psi_{x},_{x}(0,0) - F_{26}\psi_{y},_{y}(0,0) - F_{66}(\psi_{x},_{y}(0,0)+\psi_{y},_{x}(0,0))$$

$$+ (4/3h^2)H_{16}(\psi_{x},_{x}(0,0)+w,_{xx}(0,0)) + (4/3h^2)H_{26}(\psi_{y},_{y}(0,0)+w,_{yy}(0,0))$$

$$+ (4/3h^2)H_{66}(\psi_{x},_{y}(0,0)+\psi_{y},_{x}(0,0)+2w,_{xy}(0,0))$$

$$+ F_{16}\psi_{x},_{x}(0,b) + F_{26}\psi_{y},_{y}(0,b) + F_{66}(\psi_{x},_{y}(0,b)+\psi_{y},_{x}(0,b))$$

$$- (4/3h^2)H_{16}(\psi_{x},_{x}(0,b)+w,_{xx}(0,b)) - (4/3h^2)H_{26}(\psi_{y},_{y}(0,b)+w,_{yy}(0,b))$$

$$- (4/3h^2)H_{66}(\psi_{x},_{y}(0,b)+\psi_{y},_{x}(0,b)+2w,_{xy}(0,b)) \Big\} \delta w = 0 \qquad (72)$$

$\delta\psi_x$:

$$\int_0^b\int_0^a \Big\{ D_{11}\psi_{x},_{xx} + D_{12}\psi_{y},_{xy} + D_{16}(\psi_{x},_{xy}+\psi_{y},_{xx})$$

$$- (4/3h^2)F_{11}(\psi_{x},_{xx}+w,_{xxx}) - (4/3h^2)F_{12}(\psi_{y},_{xy}+w,_{xyy})$$

$$- (4/3h^2)F_{16}(\psi_{x},_{xy}+\psi_{y},_{xx}+2w,_{xxy}) + D_{16}\psi_{x},_{xy} + D_{26}\psi_{y},_{yy}$$

$$+ D_{66}(\psi_{x},_{yy}+\psi_{y},_{xy}) - (4/3h^2)F_{16}(\psi_{x},_{xy}+w,_{xxy}) - (4/3h^2)F_{26}(\psi_{y},_{yy}+w,_{yyy})$$

$$- (4/3h^2)F_{66}(\psi_{x},_{yy}+\psi_{y},_{xy}+2w,_{xyy}) - (A_{45}-(4/h^2)D_{45})(\psi_y+w,_{y})$$

$$- (A_{55}-(4/h^2)D_{55})(\psi_x+w,_{x}) + (4/h^2)(D_{45}-(4/h^2)F_{45})(\psi_y+w,_{y})$$

$$+ (4/h^2)(D_{55}-(4/h^2)F_{55})(\psi_x+w,_{x}) - (4/3h^2)F_{11}\psi_{x},_{xx} - (4/3h^2) F_{12}\psi_{y},_{xy}$$

$$- (4/3h^2)F_{16}(\psi_{x},_{xy}+\psi_{y},_{xx}) + (16/9h^4)H_{11}(\psi_{x},_{xx}+w,_{xxx})$$

$$+ (16/9h^4)H_{12}(\psi_{y},_{xy}+w,_{xyy}) + (16/9h^4)H_{16}(\psi_{x},_{xy}+\psi_{y},_{xx}+2w,_{xxy})$$

$$- (4/3h^2)F_{16}\psi_{x},_{xy} - (4/3h^2) F_{26}\psi_{y},_{yy} - (4/3h^2)F_{66}(\psi_{x},_{yy}+\psi_{y},_{xy})$$

$$+ (16/9h^4)H_{16}(\psi_{x},_{xy}+w,_{xxy}) + (16/9h^4)H_{26}(\psi_{y},_{yy}+w,_{yyy})$$

$$+ (16/9h^4)H_{66}(\psi_{x},_{yy}+\psi_{y},_{xy}+2w,_{xyy}) + \omega^2\bar{I}_3\psi_x - \omega^2(4/3h^2)\bar{I}_5 w,_{x} \Big\} \delta\psi_x \, dxdy$$

$$+ \int_0^b \Big\{ D_{11}\psi_{x},_{x}(0,y) + D_{12}\psi_{y},_{y}(0,y) + D_{16}(\psi_{x},_{y}(0,y)+\psi_{y},_{x}(0,y))$$

$$- (4/3h^2)F_{11}(\psi_{x},_{x}(0,y)+w,_{xx}(0,y)) - (4/3h^2)F_{12}(\psi_{y},_{y}(0,y)+w,_{yy}(0,y))$$

$$- (4/3h^2)F_{16}(\psi_{x,y}(0,y)+\psi_{y,x}(0,y)+2w_{,xy}(0,y)) - (8/3h^2)F_{11}\psi_{x,x}(0,y)$$

$$- (8/3h^2)F_{12}\psi_{y,y}(0,y) - (8/3h^2)F_{16}(\psi_{x,y}(0,y)+\psi_{y,x}(0,y))$$

$$+ (32/9h^4)H_{11}(\psi_{x,x}(0,y)+w_{,xx}(0,y)) + (32/9h^4)H_{12}(\psi_{y,y}(0,y)+w_{,yy}(0,y))$$

$$+ (32/9h^4)H_{16}(\psi_{x,y}(0,y)+\psi_{y,x}(0,y)+2w_{,xy}(0,y))$$

$$- D_{11}\psi_{x,x}(a,y) - D_{12}\psi_{y,y}(a,y) - D_{16}(\psi_{x,y}(a,y)+\psi_{y,x}(a,y))$$

$$+ (4/3h^2)F_{11}(\psi_{x,x}(a,y)+w_{,xx}(a,y)) + (4/3h^2)F_{12}(\psi_{y,y}(a,y)+w_{,yy}(a,y))$$

$$+ (4/3h^2)F_{16}(\psi_{x,y}(a,y)+\psi_{y,x}(a,y)+2w_{,xy}(a,y)) + (8/3h^2)F_{11}\psi_{x,x}(a,y)$$

$$+ (8/3h^2)F_{12}\psi_{y,y}(a,y) + (8/3h^2)F_{16}(\psi_{x,y}(a,y)+\psi_{y,x}(a,y))$$

$$- (32/9h^4)H_{11}(\psi_{x,x}(a,y)+w_{,xx}(a,y)) - (32/9h^4)H_{12}(\psi_{y,y}(a,y)+w_{,yy}(a,y))$$

$$- (32/9h^4)H_{16}(\psi_{x,y}(a,y)+\psi_{y,x}(a,y)+2w_{,xy}(a,y)) \Big\} \psi_x \, dy$$

$$+ \int_0^a \Big\{ D_{16}\psi_{x,x}(x,0) + D_{26}\psi_{y,y}(x,0) + D_{66}(\psi_{x,y}(x,0)+\psi_{y,x}(x,0))$$

$$- (4/3h^2)F_{16}(\psi_{x,x}(x,0)+w_{,xx}(x,0)) - (4/3h^2)F_{26}(\psi_{y,y}(x,0)+w_{,yy}(x,0))$$

$$- (4/3h^2)F_{66}(\psi_{x,y}(x,0)+\psi_{y,x}(x,0)+2w_{,xy}(x,0)) - (4/3h^2)F_{16}\psi_{x,x}(x,0)$$

$$- (4/3h^2)F_{26}\psi_{y,y}(x,0) - (4/3h^2)F_{66}(\psi_{x,y}(x,0)+\psi_{y,x}(x,0))$$

$$+ (16/9h^4)H_{16}(\psi_{x,x}(x,0)+w_{,xx}(x,0)) + (16/9h^4)H_{26}(\psi_{y,y}(x,0)+w_{,yy}(x,0))$$

$$+ (16/9h^9)H_{66}(\psi_{x,y}(x,0)+\psi_{y,x}(x,0)+2w_{,xy}(x,0))$$

$$- D_{16}\psi_{x,x}(x,b) - D_{26}\psi_{y,y}(x,b) - D_{66}(\psi_{x,y}(x,b)+\psi_{y,x}(x,b))$$

$$+ (4/3h^2)F_{16}(\psi_x,_x(x,b)+w,_{xx}(x,b)) + (4/3h^2)F_{26}(\psi_y,_y(x,b)+w,_{yy}(x,b))$$

$$+ (4/3h^2)F_{66}(\psi_x,_y(x,b)+\psi_y,_x(x,b)+2w,_{xy}(x,b)) + (4/3h^2)F_{16}\psi_x,_x(x,b)$$

$$+ (4/3h^2)F_{26}\psi_y,_y(x,b) + (4/3h^2)F_{66}(\psi_x,_y(x,b)+\psi_y,_x(x,b))$$

$$- (16/9h^4)H_{16}(\psi_x,_x(x,b)+w,_{xx}(x,b)) - (16/9h^4)H_{26}(\psi_y,_y(x,b)+w,_{yy}(x,b))$$

$$- (16/9h^9)H_{66}(\psi_x,_y(x,b)+\psi_y,_x(x,b)+2w,_{xy}(x,b)) \Big\} \delta\psi_x \, dx = 0 \qquad (73)$$

$\delta\psi_y$:

$$\int_0^b\int_0^a \Big\{ D_{12}\psi_x,_{xy} + D_{22}\psi_y,_{yy} + D_{26}(\psi_x,_{yy}+\psi_y,_{xy})$$

$$- (4/3h^2)F_{12}(\psi_x,_{xy}+w,_{xxy}) - (4/3h^2)F_{22}(\psi_y,_{yy}+w,_{yyy})$$

$$- (4/3h^2)F_{26}(\psi_x,_{yy}+\psi_y,_{xy}+2w,_{xyy}) + D_{16}\psi_x,_{xx} + D_{26}\psi_y,_{xy} + D_{66}(\psi_x,_{xy}+\psi_y,_{xx})$$

$$- (4/3h^2)F_{16}(\psi_x,_{xx}+w,_{xxx}) - (4/3h^2)F_{66}(\psi_x,_{xy}+\psi_y,_{xx}+2w,_{xxy})$$

$$- (A_{44}-(4/h^2)D_{44})(\psi_y+w,_y) - (A_{45}-(4/h^2)D_{45})(\psi_x+w,_x)$$

$$+ (4/h^2)(D_{44}-(4/h^2)F_{44})(\psi_y+w,_y) + (4/h^2)(D_{45}-(4/h^2)F_{45})(\psi_x+w,_x)$$

$$- (4/3h^2)F_{12}\psi_x,_{xy} - (4/3h^2)F_{22}\psi_y,_{yy} - (4/3h^2)F_{26}(\psi_x,_{yy}+\psi_y,_{xy})$$

$$+ (16/9h^4)H_{12}(\psi_x,_{xy}+w,_{xxy}) + (16/9h^4)H_{22}(\psi_y,_{yy}+w,_{yyy})$$

$$+ (16/9h^4)H_{26}(\psi_x,_{yy}+\psi_y,_{xy}+2w,_{xyy}) - (4/3h^2)F_{16}\psi_x,_{xx} - (4/3h^2)F_{26}\psi_y,_{xy}$$

$$- (4/3h^2)F_{66}(\psi_x,_{xy}+\psi_y,_{xx}) + (16/9h^4)H_{16}(\psi_x,_{xx}+w,_{xxx})$$

$$+ (16/9h^4)H_{26}(\psi_{y},_{xy}+w,_{xyy}) + (16/9h^4)H_{66}(\psi_{x},_{xy}+\psi_{y},_{xx}+2w,_{xxy})$$

$$+ \omega^2\bar{I}_3\psi_y - \omega^2(4/3h^2)\bar{I}_5 w,_y \Big\} \delta\psi_y \, dx\, dy$$

$$+ \int_0^b \Big\{ D_{16}\psi_{x},_{x}(0,y) + D_{26}\psi_{y},_{y}(0,y) + D_{66}(\psi_{x},_{y}(0,y)+\psi_{y},_{x}(0,y))$$

$$- (4/3h^2)F_{16}(\psi_{x},_{x}(0,y)+w,_{xx}(0,y)) - (4/3h^2)F_{26}(\psi_{y},_{y}(0,y)+w,_{yy}(0,y))$$

$$- (4/3h^2)F_{66}(\psi_{x},_{y}(0,y)+\psi_{y},_{x}(0,y)+2w,_{xy}(0,y)) - (4/3h^2)F_{16}\psi_{x},_{x}(0,y)$$

$$- (4/3h^2)F_{26}\psi_{y},_{y}(0,y) - (4/3h^2)F_{66}(\psi_{x},_{y}(0,y)+\psi_{y},_{x}(0,y))$$

$$+ (16/9h^4)H_{16}(\psi_{x},_{x}(0,y)+w,_{xx}(0,y)) + (16/9h^4)H_{26}(\psi_{y},_{y}(0,y)+w,_{yy}(0,y))$$

$$+ (16/9h^4)H_{66}(\psi_{x},_{y}(0,y)+\psi_{y},_{x}(0,y)+2w,_{xy}(0,y))$$

$$- D_{16}\psi_{x},_{x}(a,y) - D_{26}\psi_{y},_{y}(a,y) - D_{66}(\psi_{x},_{y}(a,y)+\psi_{y},_{x}(a,y))$$

$$+ (4/3h^2)F_{16}(\psi_{x},_{x}(a,y)+w,_{xx}(a,y)) + (4/3h^2)F_{26}(\psi_{y},_{y}(a,y)+w,_{yy}(a,y))$$

$$+ (4/3h^2)F_{66}(\psi_{x},_{y}(a,y)+\psi_{y},_{x}(a,y)+2w,_{xy}(a,y)) + (4/3h^2)F_{16}\psi_{x},_{x}(a,y)$$

$$+ (4/3h^2)F_{26}\psi_{y},_{y}(a,y) + (4/3h^2)F_{66}(\psi_{x},_{y}(a,y)+\psi_{y},_{x}(a,y))$$

$$- (16/9h^4)H_{16}(\psi_{x},_{x}(a,y)+w,_{xx}(a,y)) - (16/9h^4)H_{26}(\psi_{y},_{y}(a,y)+w,_{yy}(a,y))$$

$$- (16/9h^4)H_{66}(\psi_{x},_{y}(a,y)+\psi_{y},_{x}(a,y)+2w,_{xy}(a,y)) \Big\} \delta\psi_y \, dy$$

$$+ \int_0^a \Big\{ D_{12}\psi_{x},_{x}(x,0) + D_{22}\psi_{y},_{y}(x,0) + D_{26}(\psi_{x},_{y}(x,0)+\psi_{y},_{x}(x,0))$$

$$- (4/3h^2)F_{12}(\psi_{x},_{x}(x,0)+w,_{xx}(x,0)) - (4/3h^2)F_{22}(\psi_{y},_{y}(x,0)+w,_{yy}(x,0))$$

$$- (4/3h^2)F_{26}(\psi_{x},_{y}(x,0)+\psi_{y},_{x}(x,0)+2w,_{xy}(x,0)) - (8/3h^2)F_{12}\psi_{x},_{x}(x,0)$$

$$- (8/3h^2)F_{22}\psi_{y},_{y}(x,0) - (8/3h^2)F_{26}(\psi_{x},_{y}(x,0)+\psi_{y},_{x}(x,0))$$

$$+ (32/9h^4)H_{12}(\psi_{x},_{x}(x,0)+w,_{xx}(x,0)) + (32/9h^4)H_{22}(\psi_{y},_{y}(x,0)+w,_{yy}(x,0))$$

$$+ (32/9h^4)H_{26}(\psi_{x},_{y}(x,0)+\psi_{y},_{x}(x,0)+2w,_{xy}(x,0))$$

$$- D_{12}\psi_{x},_{x}(x,b) - D_{22}\psi_{y},_{y}(x,b) - D_{26}(\psi_{x},_{y}(x,b)+\psi_{y},_{x}(x,b))$$

$$+ (4/3h^2)F_{12}(\psi_{x},_{x}(x,b)+w,_{xx}(x,b)) + (4/3h^2)F_{22}(\psi_{y},_{y}(x,b)+w,_{yy}(x,b))$$

$$+ (4/3h^2)F_{26}(\psi_{x},_{y}(x,b)+\psi_{y},_{x}(x,b)+2w,_{xy}(x,b)) + (8/3h^2)F_{12}\psi_{x},_{x}(x,b)$$

$$+ (8/3h^2)F_{22}\psi_{y},_{y}(x,b) + (8/3h^2)F_{26}(\psi_{x},_{y}(x,b)+\psi_{y},_{x}(x,b))$$

$$- (32/9h^4)H_{12}(\psi_{x},_{x}(x,b)+w,_{xx}(x,b)) - (32/9h^4)H_{22}(\psi_{y},_{y}(x,b)+w,_{yy}(x,b))$$

$$- (32/9h^4)H_{26}(\psi_{x},_{y}(x,b)+\psi_{y},_{x}(x,b)+2w,_{xy}(x,b)) \Big\} \delta\psi_{y}\ dx = 0 \qquad (74)$$

We now nondimensionalize Eqs.(72), (73), and (74) by using the following definitions

$$\left.\begin{aligned} a_{ij} &= A_{ij} / E_2 h \\ d_{ij} &= D_{ij} / E_2 h^3 \\ f_{ij} &= F_{ij} / E_2 h^5 \\ h_{ij} &= H_{ij} / E_2 h^7 \end{aligned}\right\} \quad \text{normalized stiffnesses}$$

where E_2 is the transverse modulus

$$R = a / b \quad \Big\} \quad \text{aspect ratio}$$

$$s = a / h \quad \Big\} \quad \text{span-to-depth ratio}$$

$$\bar{w} = w / h \quad \Big\} \quad \text{normalized out-of-plane displacement} \tag{75}$$

$$\left.\begin{aligned} \xi &= x / a \\ \eta &= y / b \end{aligned}\right\} \quad \text{normalized coordinates}$$

$$\bar{\omega}^2 = a^2 \omega^2 p / E_2 h \quad \Big\} \quad \text{normalized natural frequency}$$

$$\bar{N}_0 = N_0 a^2 / E_2 h^3 \quad \Big\} \quad \text{normalized buckling load}$$

$$p = \sum_{k=1}^{N} \rho_k t_k \quad \Big\} \quad \text{laminate density}$$

where ρ_k = ply density, t_k = ply thickness, and N = number of plys. Note that $p = \rho h$ for non-hybrid plates. This work will not consider

hybrid plates, therefore, the inertia terms may be simplified as follows using Eqs.(33)

$$I_1 = p \quad , \quad I_3 = ph^2/\ 12 \quad , \quad I_5 = ph^4/\ 80 \quad , \quad I_7 = ph^6/\ 448$$
$$\bar{I}_3 = 17ph^2/\ 315 \quad , \quad \bar{I}_5 = ph^4/\ 105 \tag{76}$$

The inplane loads may be simplified by using the concept of proportional loading. Proportional loading assumes that the axial and shear components are multiples of a unit load N_0. Thus

$$\bar{N}_1 = -k_1N_0 \quad , \quad \bar{N}_2 = -k_2N_0 \quad , \quad \bar{N}_6 = k_3N_0 \tag{77}$$

where k_1, k_2, and k_3 are coefficients chosen for a given loading condition. A negative sign is used in front of k_1 and k_2 because $\bar{N}_1$ and $\bar{N}_2$ are compressive loads.

We now carry out the normalization process by multiplying the δw equation of motion (Eq.(72)) by $a^4/\ E_2h^4$ and the $\delta\psi_x$ and $\delta\psi_y$ equations of motion (Eqs.(73) and (74)) by $a^4/\ E_2h^5$. The resulting equations can provide natural frequency solutions by setting the proportional load coefficients to zero or buckling solutions by setting the plate inertias to zero. The final equations of motion are

δw :

$$\int_0^1\int_0^1 \Bigg\{ (\ a_{45}-8d_{45}+16f_{45}\ )(s^3\psi_{\eta},_{\xi}+s^3R\psi_{\xi},_{\eta}+2s^2R\bar{w},_{\xi\eta})$$

$$+ (a_{55}-8d_{55}+16f_{55})(s^3\Psi_{\xi},_{\xi}+s^2\bar{w},_{\xi\xi})$$

$$+ (a_{44}-8d_{44}+16f_{44})(s^3R\Psi_{\eta},_{\eta}+s^2R^2\bar{w},_{\eta\eta})$$

$$+ \bar{N}_0(-k_1\bar{w},_{\xi\xi} - R^2k_2\bar{w},_{\eta\eta} + Rk_3\bar{w},_{\xi\eta})$$

$$+ (4/3)f_{11}s\Psi_{\xi},_{\xi\xi\xi}+ (4/3)sRf_{12}\Psi_{\eta},_{\xi\xi\eta} + (4/3)f_{16}(sR\Psi_{\xi},_{\xi\xi\eta}+s\Psi_{\eta},_{\xi\xi\xi})$$

$$- (16/9)h_{11}(s\Psi_{\xi},_{\xi\xi\xi}+\bar{w},_{\xi\xi\xi\xi}) - (16/9)h_{12}(sR\Psi_{\eta},_{\xi\xi\eta}+R^2\bar{w},_{\xi\xi\eta\eta})$$

$$- (16/9)h_{16}(sR\Psi_{\xi},_{\xi\xi\eta}+s\Psi_{\eta},_{\xi\xi\xi}+2R\bar{w},_{\xi\xi\xi\eta}) + (4/3)sR^2f_{12}\Psi_{\xi},_{\xi\eta\eta}$$

$$+ (4/3)sR^3f_{22}\Psi_{\eta},_{\eta\eta\eta} + (4/3)f_{26}(sR^3\Psi_{\xi},_{\eta\eta\eta}+sR^2\Psi_{\eta},_{\xi\eta\eta})$$

$$- (16/9)h_{12}(sR^2\Psi_{\xi},_{\xi\eta\eta}+R^2\bar{w},_{\xi\xi\eta\eta}) - (16/9)h_{22}(sR^3\Psi_{\eta},_{\eta\eta\eta}+R^4\bar{w},_{\eta\eta\eta\eta})$$

$$- (16/9)h_{26}(sR^3\Psi_{\xi},_{\eta\eta\eta}+sR^2\Psi_{\eta},_{\xi\eta\eta}+2R^3\bar{w},_{\xi\eta\eta\eta}) + (8/3)sRf_{16}\Psi_{\xi},_{\xi\xi\eta}$$

$$+ (8/3)sR^2f_{26}\Psi_{\eta},_{\xi\eta\eta} + (8/3)f_{66}(sR^2\Psi_{\xi},_{\xi\eta\eta}+sR\Psi_{\eta},_{\xi\xi\eta})$$

$$- (32/9)h_{16}(sR\Psi_{\xi},_{\xi\xi\eta}+R\bar{w},_{\xi\xi\xi\eta}) - (32/9)h_{26}(sR^2\Psi_{\eta},_{\xi\eta\eta}+R^3\bar{w},_{\xi\eta\eta\eta})$$

$$- (32/9)h_{66}(sR^2\Psi_{\xi},_{\xi\eta\eta}+sR\Psi_{\eta},_{\xi\xi\eta}+2R^2\bar{w},_{\xi\xi\eta\eta})$$

$$\left. + \omega^2 (s^2\bar{w}-16/4032\bar{w},_{\xi\xi}-16/4032R^2\bar{w},_{\eta\eta}+4/315s\Psi_{\xi},_{\xi}+4/315sR\Psi_{\eta},_{\eta}) \right\} \delta w d\xi d\eta$$

$$+ \int_0^1 \left\{ \bar{N}_0(-k_1\bar{w},_{\xi}(0,\eta) + k_3R\bar{w},_{\eta}(0,\eta) + (a_{55}-8d_{55}+16f_{55}) \right.$$

$$(s^3\Psi_{\xi}(0,\eta)+s^2\bar{w},_{\xi}(0,\eta)) + (a_{45}-8d_{45}+16f_{45})(s^3\Psi_{\eta}(0,\eta)+s^2R\bar{w},_{\eta}(0,\eta))$$

$$+ (4/3)sf_{11}\Psi_{\xi},_{\xi\xi}(0,\eta) + (4/3)sRf_{12}\Psi_{\eta},_{\xi\eta}(0,\eta) + (4/3)f_{16}$$

$$(sR\Psi_{\xi},_{\xi\eta}(0,\eta)+s\Psi_{\eta},_{\xi\xi}(0,\eta)) - (16/9)h_{11}(s\Psi_{\xi},_{\xi\xi}(0,\eta)+\bar{w},_{\xi\xi\xi}(0,\eta))$$

$$- (16/9)h_{12}(sR\psi_{\eta,\xi\eta}(0,\eta)+R^2\bar{w}_{,\xi\eta\eta}(0,\eta)) - (16/9)h_{16}$$

$$(sR\psi_{\xi,\xi\eta}(0,\eta)+s\psi_{\eta,\xi\xi}(0,\eta)+2R\bar{w}_{,\xi\xi\eta}(0,\eta)) + (8/3)sRf_{16}\psi_{\xi,\xi\eta}(0,\eta)$$

$$+ (8/3)sR^2f_{26}\psi_{\eta,\eta\eta}(0,\eta) + (8/3)f_{66}(sR^2\psi_{\xi,\eta\eta}(0,\eta)+sR\psi_{\eta,\xi\eta}(0,\eta))$$

$$- (32/9)h_{16}(sR\psi_{\xi,\xi\eta}(0,\eta)+R\bar{w}_{,\xi\xi\eta}(0,\eta)) - (32/9)h_{26}(sR^2\psi_{\eta,\eta\eta}(0,\eta)$$

$$+R^3\bar{w}_{,\eta\eta\eta}(0,\eta)) - (32/9)h_{66}(sR^2\psi_{\xi,\eta\eta}(0,\eta)+sR\psi_{\eta,\xi\eta}(0,\eta)+2R^2\bar{w}_{,\xi\eta\eta}(0,\eta))$$

$$-\bar{N}_0(-k_1\bar{w}_{,\xi}(1,\eta) + k_3\bar{w}_{,\eta}(1,\eta)) - (a_{55}-8d_{55}+16f_{55})$$

$$(s^3\psi_\xi(1,\eta)+s^2\bar{w}_{,\xi}(1,\eta)) - (a_{45}-8d_{45}+16f_{45})(s^3\psi_\eta(1,\eta)+s^2R\bar{w}_{,\eta}(1,\eta))$$

$$- (4/3)sf_{11}\psi_{\xi,\xi\xi}(1,\eta) - (4/3)sRf_{12}\psi_{\eta,\xi\eta}(1,\eta) - (4/3)f_{16}$$

$$(sR\psi_{\xi,\xi\eta}(1,\eta)+s\psi_{\eta,\xi\xi}(1,\eta)) + (16/9)h_{11}(s\psi_{\xi,\xi\xi}(1,\eta)+\bar{w}_{,\xi\xi\xi}(1,\eta))$$

$$+ (16/9)h_{12}(sR\psi_{\eta,\xi\eta}(1,\eta)+sR^2\bar{w}_{,\xi\eta\eta}(1,\eta)) + (16/9)h_{16}$$

$$(sR\psi_{\xi,\xi\eta}(1,\eta)+s\psi_{\eta,\xi\xi}(1,\eta)+2R\bar{w}_{,\xi\xi\eta}(1,\eta)) - (8/3)sRf_{16}\psi_{\xi,\xi\eta}(1,\eta)$$

$$- (8/3)sR^2f_{26}\psi_{\eta,\eta\eta}(1,\eta) - (8/3)f_{66}(sR^2\psi_{\xi,\eta\eta}(1,\eta)+sR\psi_{\eta,\xi\eta}(1,\eta))$$

$$+ (32/9)h_{16}(sR\psi_{\xi,\xi\eta}(1,\eta)+R\bar{w}_{,\xi\xi\eta}(1,\eta)) + (32/9)h_{26}(sR^2\psi_{\eta,\eta\eta}(1,\eta)+R^3\bar{w}_{,\eta\eta\eta}$$

$$(1,\eta)) + (32/9)h_{66}(sR^2\psi_{\xi,\eta\eta}(1,\eta)+sR\psi_{\eta,\xi\eta}(1,\eta)+2R^2\bar{w}_{,\xi\eta\eta}(1,\eta)) \Big\} \delta w \, d\eta$$

$$+ \int_0^1 \Big\{ \bar{N}_0(-k_2\bar{w}_{,\eta}(\xi,0) + k_3\bar{w}_{,\xi}(\xi,0) + (a_{44}-8d_{44}+16f_{44})$$

$$(s^3\psi_\eta(\xi,0)+s^2R\bar{w}_{,\eta}(\xi,0)) + (a_{45}-8d_{45}+16f_{45})(s^3\psi_\xi(\xi,0)+s^2\bar{w}_{,\xi}(\xi,0))$$

$$- (4/3)sRf_{12}\psi_{\xi,\xi\eta}(\xi,0) - (4/3)sR^2f_{22}\psi_{\eta,\eta\eta}(\xi,0) - (4/3)f_{26}$$

$$(sR^2\psi_{\xi},_{\eta\eta}(\xi,0)+sR\psi_{\eta},_{\xi\eta}(\xi,0)) + (16/9)h_{12}(sR\psi_{\xi},_{\xi\eta}(\xi,0)+R\bar{w},_{\xi\xi\eta}(\xi,0))$$

$$+ (16/9)h_{22}(sR^2\psi_{\eta},_{\eta\eta}(\xi,0)+R^3\bar{w},_{\eta\eta\eta}(\xi,0)) + (16/9)h_{26}$$

$$(sR^2\psi_{\xi},_{\eta\eta}(\xi,0)+sR\psi_{\eta},_{\xi\eta}(\xi,0)+2R^2\bar{w},_{\xi\eta\eta}(\xi,0)) - (8/3)sf_{16}\psi_{\xi},_{\xi\xi}(\xi,0)$$

$$- (8/3)srf_{26}\psi_{\eta},_{\xi\eta}(\xi,0) - (8/3)f_{66}(sR\psi_{\xi},_{\xi\eta}(\xi,0)+s\psi_{\eta},_{\xi\xi}(\xi,0))$$

$$+ (32/9)h_{16}(s\psi_{\xi},_{\xi\xi}(\xi,0)+\bar{w},_{\xi\xi\xi}(\xi,0)) + (32/9)h_{26}(sR\psi_{\eta},_{\xi\eta}(\xi,0)$$

$$+R^2w,_{\xi\eta\eta}(\xi,0)) + (32/9)h_{66}(sR\psi_{\xi},_{\xi\eta}(\xi,0)+s\psi_{\eta},_{\xi\xi}(\xi,0)+2R\bar{w},_{\xi\xi\eta}(\xi,0))$$

$$-\bar{N}_0(-k_2R\bar{w},_{\eta}(\xi,1)+k_3\bar{w},_{\xi}(\xi,1)) - (\ a_{44}-8d_{44}+16f_{44}\ )$$

$$(s^3\psi_{\eta}(\xi,1)+s^2R\bar{w},_{\eta}(\xi,1)) - (\ a_{45}-8d_{45}+16f_{45}\ )(s^3\psi_{\xi}(\xi,1)+s^3\bar{w},_{\xi}(\xi,1))$$

$$+ (4/3)sRf_{12}\psi_{\xi},_{\xi\eta}(\xi,1) + (4/3)sR^2f_{22}\psi_{\eta},_{\eta\eta}(\xi,1) + (4/3)f_{26}$$

$$(sR^2\psi_{\xi},_{\eta\eta}(\xi,1)+sR\psi_{\eta},_{\xi\eta}(\xi,1)) - (16/9)h_{12}(sR\psi_{\xi},_{\xi\eta}(\xi,1)+R\bar{w},_{\xi\xi\eta}(\xi,1))$$

$$- (16/9)h_{22}(sR^2\psi_{\eta},_{\eta\eta}(\xi,1)+R^3\bar{w},_{\eta\eta\eta}(\xi,1)) - (16/9)h_{26}$$

$$(sR^2\psi_{\xi},_{\eta\eta}(\xi,1)+sR\psi_{\eta},_{\xi\eta}(\xi,1)+2R^2\bar{w},_{\xi\eta\eta}(\xi,1)) + (8/3)f_{16}s\psi_{\xi},_{\xi\xi}(\xi,1)$$

$$+ (8/3)f_{26}sR\psi_{\eta},_{\xi\eta}(\xi,1) + (8/3)f_{66}(sR\psi_{\xi},_{\xi\eta}(\xi,1)+s\psi_{\eta},_{\xi\xi}(\xi,1))$$

$$- (32/9)h_{16}(s\psi_{\xi},_{\xi\xi}(\xi,1)+\bar{w},_{\xi\xi\xi}(\xi,1)) - (32/9)h_{26}$$

$$(sR\psi_{\eta},_{\xi\eta}(\xi,1)+R^2\bar{w},_{\xi\eta\eta}(\xi,1)) - (32/9)h_{66}$$

$$\left.(sR\psi_{\xi},_{\xi\eta}(\xi,1)+s\psi_{\eta},_{\xi\xi}(\xi,1)+2R\bar{w},_{\xi\xi\eta}(\xi,1))\ \right\}\delta w\ d\xi$$

$$- (8/3)\left\{\ f_{16}s\psi_{\xi},_{\xi}(1,0) + f_{26}sR\psi_{\eta},_{\eta}(1,0) + f_{66}(sR\psi_{\xi},_{\eta}(1,0)+s\psi_{\eta},_{\xi}(1,0))\right.$$

$$- (4/3)h_{16}(s\Psi_{\xi},_{\xi}(1,0)+\bar{w},_{\xi\xi}(1,0)) - (4/3)h_{26}(sR\Psi_{\eta},_{\eta}(1,0)+R^2\bar{w},_{\eta\eta}(1,0))$$

$$- (4/3)h_{66}(sR\Psi_{\xi},_{\eta}(1,0)+s\Psi_{\eta},_{\xi}(1,0)+2R\bar{w},_{\xi\eta}(1,0))$$

$$- f_{16}s\Psi_{\xi},_{\xi}(1,1) - f_{26}sR\Psi_{\eta},_{\eta}(1,1) - f_{66}(sR\Psi_{\xi},_{\eta}(1,1)+s\Psi_{\eta},_{\xi}(1,1))$$

$$+ (4/3)h_{16}(s\Psi_{\xi},_{\xi}(1,1)+\bar{w},_{\xi\xi}(1,1)) + (4/3)h_{26}(sR\Psi_{\eta},_{\eta}(1,1)+R^2\bar{w},_{\eta\eta}(1,1))$$

$$+ (4/3)h_{66}(sR\Psi_{\xi},_{\eta}(1,1)+s\Psi_{\eta},_{\xi}(1,1)+2R\bar{w},_{\xi\eta}(1,1))$$

$$- f_{16}s\Psi_{\xi},_{\xi}(0,0) - f_{26}sR\Psi_{\eta},_{\eta}(0,0) - f_{66}(sR\Psi_{\xi},_{\eta}(0,0)+s\Psi_{\eta},_{\xi}(0,0))$$

$$+ (4/3)h_{16}(s\Psi_{\xi},_{\xi}(0,0)+\bar{w},_{\xi\xi}(0,0)) + (4/3)h_{26}(sR\Psi_{\eta},_{\eta}(0,0)+R^2\bar{w},_{\eta\eta}(0,0))$$

$$+ (4/3)h_{66}(sR\Psi_{\xi},_{\eta}(0,0)+s\Psi_{\eta},_{\xi}(0,0)+2R\bar{w},_{\xi\eta}(0,0))$$

$$+ f_{16}s\Psi_{\xi},_{\xi}(0,1) + f_{26}sR\Psi_{\eta},_{\eta}(0,1) + f_{66}(sR\Psi_{\xi},_{\eta}(0,1)+s\Psi_{\eta},_{\xi}(0,1))$$

$$- (4/3)h_{16}(s\Psi_{\xi},_{\xi}(0,1)+\bar{w},_{\xi\xi}(0,1)) - (4/3)h_{26}(sR\Psi_{\eta},_{\eta}(0,1)+R^2\bar{w},_{\eta\eta}(0,1))$$

$$- (4/3)h_{66}(sR\Psi_{\xi},_{\eta}(0,1)+s\Psi_{\eta},_{\xi}(0,1)+2R\bar{w},_{\xi\eta}(0,1)) \Big\} \delta w = 0 \qquad (78)$$

$\delta\Psi_{\xi}$:

$$\int_0^1\int_0^1 \Big\{ d_{11}s^2\Psi_{\xi},_{\xi\xi} + d_{12}s^2R\Psi_{\eta},_{\xi\eta} + d_{16}(s^2R\Psi_{\xi},_{\xi\eta}+s^2\Psi_{\eta},_{\xi\xi})$$

$$- (4/3)f_{11}(s^2\Psi_{\xi},_{\xi\xi}+s\bar{w},_{\xi\xi\xi}) - (4/3)f_{12}(s^2R\Psi_{\eta},_{\xi\eta}+sR^2\bar{w},_{\xi\eta\eta})$$

$$- (4/3)f_{16}(s^2R\Psi_{\xi},_{\xi\eta}+s^2\Psi_{\eta},_{\xi\xi}+2sR\bar{w},_{\xi\xi\eta}) + d_{16}s^2R\Psi_{\xi},_{\xi\eta} + d_{26}s^2R^2\Psi_{\eta},_{\eta\eta}$$

$$+ d_{66}(s^2R^2\Psi_{\xi},_{\eta\eta}+s^2R\Psi_{\eta},_{\xi\eta}) - (4/3)f_{16}(s^2R\Psi_{\xi},_{\xi\eta}+sR\bar{w},_{\xi\xi\eta})$$

$$- (4/3)f_{26}(s^2R^2\psi_{\eta},_{\eta\eta}+sR^3\bar{w},_{\eta\eta\eta}) - (4/3)f_{66}(s^2R^2\psi_{\xi},_{\eta\eta}+s^2R\psi_{\eta},_{\xi\eta}+2sR^2\bar{w},_{\xi\eta\eta})$$

$$- (a_{45}-8d_{45}+16f_{45})(s^4\psi_{\eta}+s^3R\bar{w},_{\eta}) - (a_{55}-8d_{55}+16f_{55})$$

$$(s^4\psi_{\xi}+s^3\bar{w},_{\xi}) - (4/3)f_{11}s^2\psi_{\xi},_{\xi\xi} - (4/3)\ f_{12}s^2R\psi_{\eta},_{\xi\eta}$$

$$- (4/3)f_{16}(s^2R\psi_{\xi},_{\xi\eta}+s^2\psi_{\eta},_{\xi\xi}) + (16/9)h_{11}(s^2\psi_{\xi},_{\xi\xi}+s\bar{w},_{\xi\xi\xi})$$

$$+ (16/9)h_{12}(s^2R\psi_{\eta},_{\xi\eta}+sR^2\bar{w},_{\xi\eta\eta}) + (16/9)h_{16}(s^2R\psi_{\xi},_{\xi\eta}+s^2\psi_{\eta},_{\xi\xi}+2sR\bar{w},_{\xi\xi\eta})$$

$$- (4/3)f_{16}s^2R\psi_{\xi},_{\xi\eta} - (4/3)f_{26}s^2R^2\psi_{\eta},_{\eta\eta} - (4/3)f_{66}(s^2R^2\psi_{\xi},_{\eta\eta}+s^2R\psi_{\eta},_{\xi\eta})$$

$$+ (16/9)h_{16}(s^2R\psi_{\xi},_{\xi\eta}+sR\bar{w},_{\xi\xi\eta}) + (16/9)h_{26}(s^2R^2\psi_{\eta},_{\eta\eta}+sR^3\bar{w},_{\eta\eta\eta})$$

$$+ (16/9)h_{66}(s^2R^2\psi_{\xi},_{\eta\eta}+s^2R\psi_{\eta},_{\xi\eta}+2sR^2w,_{\xi\eta\eta})$$

$$+ \omega^2 \left(17/315s^2\psi_x - 4/315s\bar{w},_x \right) \Big\}\ \delta\psi_x\ d\xi d\eta$$

$$+ \int_0^1 \Big\{ d_{11}s^2\psi_{\xi},_{\xi}(0,\eta) + d_{12}s^2R\psi_{\eta},_{\eta}(0,\eta) + d_{16}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta))$$

$$- (4/3)f_{11}(s^2\psi_{\xi},_{\xi}(0,\eta)+s\bar{w},_{\xi\xi}(0,\eta)) - (4/3)f_{12}(s^2R\psi_{\eta},_{\eta}(0,\eta)+sR^2\bar{w},_{\eta\eta}(0,\eta))$$

$$- (4/3)f_{16}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta)+2sR\bar{w},_{\xi\eta}(0,\eta)) - (8/3)f_{11}s^2\psi_{\xi},_{\xi}(0,\eta)$$

$$- (8/3)f_{12}s^2R\psi_{\eta},_{\eta}(0,\eta) - (8/3)f_{16}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta))$$

$$+ (32/9)h_{11}(s^2\psi_{\xi},_{\xi}(0,\eta)+s\bar{w},_{\xi\xi}(0,\eta)) + (32/9)h_{12}(s^2R\psi_{\eta},_{\eta}(0,\eta)+sR^2w,_{\eta\eta}(0,\eta))$$

$$+ (32/9)h_{16}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta)+2s^2Rw,_{\xi\eta}(0,\eta))$$

$$- d_{11}s^2\psi_{\xi},_{\xi}(1,\eta) - d_{12}s^2R\psi_{\eta},_{\eta}(1,\eta) - d_{16}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta))$$

$$+ (4/3)f_{11}(s^2\psi_{\xi},_{\xi}(1,\eta)+s\bar{w},_{\xi\xi}(1,\eta)) + (4/3)f_{12}(s^2R\psi_{\eta},_{\eta}(1,\eta)+sR^2\bar{w},_{\eta\eta}(1,\eta))$$

$$+ (4/3)f_{16}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta)+2sR\bar{w},_{\xi\eta}(1,\eta)) + (8/3)f_{11}s^2\psi_{\xi},_{\xi}(1,\eta)$$

$$+ (8/3)f_{12}s^2R\psi_{\eta},_{\eta}(1,\eta) + (8/3)f_{16}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta))$$

$$- (32/9)h_{11}(s^2\psi_{\xi},_{\xi}(1,\eta)+s\bar{w},_{\xi\xi}(1,\eta)) - (32/9)h_{12}(s^2R\psi_{\eta},_{\eta}(1,\eta)+sR^2\bar{w},_{\eta\eta}(1,\eta))$$

$$- (32/9)h_{16}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta)+2sR\bar{w},_{\xi\eta}(1,\eta)) \Big\} \psi_x \, d\eta$$

$$+ \int_0^1 \Big\{ d_{16}s^2\psi_{\xi},_{\xi}(\xi,0) + d_{26}s^2r\psi_{\eta},_{\eta}(\xi,0) + d_{66}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0))$$

$$- (4/3)f_{16}(s^2\psi_{\xi},_{\xi}(\xi,0)+s\bar{w},_{\xi\xi}(\xi,0)) - (4/3)f_{26}(s^2R\psi_{\eta},_{\eta}(\xi,0)+sR^2\bar{w},_{\eta\eta}(\xi,0))$$

$$- (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0)+2sRw,_{\xi\eta}(\xi,0)) - (4/3)f_{16}s^2\psi_{\xi},_{\xi}(\xi,0)$$

$$- (4/3)f_{26}s^2R\psi_{\eta},_{\eta}(\xi,0) - (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0))$$

$$+ (16/9)h_{16}(s^2\psi_{\xi},_{\xi}(\xi,0)+s\bar{w},_{\xi\xi}(\xi,0)) + (16/9)h_{26}(s^2R\psi_{\eta},_{\eta}(\xi,0)+sR^2\bar{w},_{\eta\eta}(\xi,0))$$

$$+ (16/9)h_{66}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0)+2sR\bar{w},_{\xi\eta}(\xi,0))$$

$$- d_{16}s^2\psi_{\xi},_{\xi}(\xi,1) - d_{26}s^2R\psi_{\eta},_{\eta}(\xi,1) - d_{66}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1))$$

$$+ (4/3)f_{16}(s^2\psi_{\xi},_{\xi}(\xi,1)+s\bar{w},_{\xi\xi}(\xi,1)) + (4/3)f_{26}(s^2R\psi_{\eta},_{\eta}(\xi,1)+sR^2\bar{w},_{\eta\eta}(\xi,1))$$

$$+ (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1)+2sR\bar{w},_{\xi\eta}(\xi,1)) + (4/3)f_{16}s^2\psi_{\xi},_{\xi}(\xi,1)$$

$$+ (4/3)f_{26}s^2R\psi_{\eta},_{\eta}(\xi,1) + (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1))$$

$$- (16/9)h_{16}(s^2\psi_{\xi},_{\xi}(\xi,1)+s\bar{w},_{\xi\xi}(\xi,1)) - (16/9)h_{26}(s^2R\psi_{\eta},_{\eta}(\xi,1)+sR^2\bar{w},_{\eta\eta}(\xi,1))$$

$$- (16/9)h_{66}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1)+2sR\bar{w},_{\xi\eta}(\xi,1)) \Big\} \delta\psi_{\xi} \, d\xi = 0 \qquad (79)$$

$\delta\psi_\eta$:

$$\int_0^1\int_0^1 \Big\{ d_{12}s^2R\psi_{\xi},_{\xi\eta} + d_{22}s^2R^2\psi_{\eta},_{\eta\eta} + d_{26}(s^2R^2\psi_{\xi},_{\eta\eta}+s^2R\psi_{\eta},_{\xi\eta})$$

$$- (4/3)f_{12}(s^2R\psi_{\xi},_{\xi\eta}+sR\bar{w},_{\xi\xi\eta}) - (4/3)f_{22}(s^2R^2\psi_{\eta},_{\eta\eta}+sR^3w,_{\eta\eta\eta})$$

$$- (4/3)f_{26}(s^2R^2\psi_{\xi},_{\eta\eta}+s^2R\psi_{\eta},_{\xi\eta}+2sR^2\bar{w},_{\xi\eta\eta}) + d_{16}s^2\psi_{\xi},_{\xi\xi} + d_{26}s^2R\psi_{\eta},_{\xi\eta}$$

$$+ d_{66}(s^2R\psi_{\xi},_{\xi\eta}+s\psi_{\eta},_{\xi\xi}) - (4/3)f_{16}(s^2\psi_{\xi},_{\xi\xi}+sw,_{\xi\xi\xi})$$

$$- (4/3)f_{26}(s^2R\psi_{\xi},_{\xi\eta}+s^2R^2\bar{w},_{\xi\eta\eta} - (4/3)f_{66}(s^2R\psi_{\xi},_{\xi\eta}+s^2\psi_{\eta},_{\xi\xi}+2s^2Rw,_{\xi\xi\eta})$$

$$- (a_{44}-8d_{44}+16f_{44})(s^4\psi_{\eta}+s^3Rw,_{\eta}) - (a_{45}-8d_{45}+16f_{45})(s^4\psi_{\xi}+s^3\bar{w},_{\xi})$$

$$- (4/3)f_{12}s^2R\psi_{\xi},_{\xi\eta} - (4/3)f_{22}s^2R^2\psi_{\eta},_{\eta\eta} - (4/3)f_{26}(s^2R^2\psi_{\xi},_{\eta\eta}+s^2R\psi_{\eta},_{\xi\eta})$$

$$\underset{a}{+} (16/9)h_{12}(s^2R\psi_{\xi},_{\xi\eta}+sR\bar{w},_{\xi\xi\eta}) + (16/9)h_{22}(s^2R^2\psi_{\eta},_{\eta\eta}+sR^3w,_{\eta\eta\eta})$$

$$+ (16/9)h_{26}(s^2R^2\psi_{\xi},_{\eta\eta}+s^2R\psi_{\eta},_{\xi\eta}+2sR^2\bar{w},_{\xi\eta\eta}) - (4/3)f_{16}s^2\psi_{\xi},_{\xi\xi}$$

$$- (4/3)f_{26}s^2R\psi_{\eta},_{\xi\eta} - (4/3)f_{66}(s^2R\psi_{\xi},_{\xi\eta}+s^2\psi_{\eta},_{\xi\xi})$$

$$+ (16/9)h_{16}(s^2\psi_{\xi},_{\xi\xi}+s\bar{w},_{\xi\xi\xi})+ (16/9)h_{26}(s^2R\psi_{\eta},_{\xi\eta}+s^2R^2w,_{\xi\eta\eta})$$

$$+ (16/9)h_{66}(s^2R\psi_{\xi},_{\xi\eta}+s^2\psi_{\eta},_{\xi\xi}+2sRw,_{\xi\xi\eta})$$

$$+ \omega^2 (17/315s^2\psi_{\eta} - 4/315s\bar{w},_{\eta}) \Big\} \delta\psi_{\eta}\, d\xi\, d\eta$$

$$+ \int_0^1 \Big\{ d_{16}s^2\psi_{\xi},_{\xi}(0,\eta) + d_{26}s^2R\psi_{\eta},_{\eta}(0,\eta) + d_{66}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta))$$

$$- (4/3)f_{16}(s^2\psi_{\xi},_{\xi}(0,\eta)+s\bar{w},_{\xi\xi}(0,\eta)) - (4/3)f_{26}(s^2R\psi_{\eta},_{\eta}(0,\eta)+sR^2\bar{w},_{\eta\eta}(0,\eta))$$

$$- (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta)+2sR\bar{w},_{\xi\eta}(0,\eta)) - (4/3)f_{16}s^2\psi_{\xi},_{\xi}(0,\eta)$$

$$- (4/3)f_{26}s^2R\psi_{\eta},_{\eta}(0,\eta) - (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta))$$

$$+ (16/9)h_{16}(s^2\psi_{\xi},_{\xi}(0,\eta)+s\bar{w},_{\xi\xi}(0,\eta)) + (16/9)h_{26}(s^2R\psi_{\eta},_{\eta}(0,\eta)+sR^2\bar{w},_{\eta\eta}(0,\eta))$$

$$+ (16/9)h_{66}(s^2R\psi_{\xi},_{\eta}(0,\eta)+s^2\psi_{\eta},_{\xi}(0,\eta)+2sR\bar{w},_{\xi\eta}(0,\eta))$$

$$- d_{16}s^2\psi_{\xi},_{\xi}(1,\eta) - d_{26}s^2R\psi_{\eta},_{\eta}(1,\eta) - d_{66}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta))$$

$$+ (4/3)f_{16}(s^2\psi_{\xi},_{\xi}(1,\eta)+s\bar{w},_{\xi\xi}(1,\eta)) + (4/3)f_{26}(s^2R\psi_{\eta},_{\eta}(1,\eta)+sR^2\bar{w},_{\eta\eta}(1,\eta))$$

$$+ (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta)+2sR\bar{w},_{\xi\eta}(1,\eta)) + (4/3)f_{16}s^2\psi_{\xi},_{\xi}(1,\eta)$$

$$+ (4/3)f_{26}s^2R\psi_{\eta},_{\eta}(1,\eta) + (4/3)f_{66}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta))$$

$$- (16/9)h_{16}(s^2\psi_{\xi},_{\xi}(1,\eta)+s\bar{w},_{\xi\xi}(1,\eta)) - (16/9)h_{26}(s^2R\psi_{\eta},_{\eta}(1,\eta)+sR^2w,_{\eta\eta}(1,\eta))$$

$$- (16/9)h_{66}(s^2R\psi_{\xi},_{\eta}(1,\eta)+s^2\psi_{\eta},_{\xi}(1,\eta)+2sR\bar{w},_{\xi\eta}(1,\eta)) \Big\} \delta\psi_{\eta}\, d\eta$$

$$+ \int_0^1 \Big\{ d_{12}s^2\psi_{\xi},_{\xi}(\xi,0) + d_{22}s^2R\psi_{\eta},_{\eta}(\xi,0) + d_{26}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0))$$

$$- (4/3)f_{12}(s^2\psi_{\xi},_{\xi}(\xi,0)+s\bar{w},_{\xi\xi}(\xi,0)) - (4/3)f_{22}(s^2R\psi_{\eta},_{\eta}(\xi,0)+sR^2\bar{w},_{\eta\eta}(\xi,0))$$

$$- (4/3)f_{26}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0)+2sR\bar{w},_{\xi\eta}(\xi,0)) - (8/3)f_{12}s^2\psi_{\xi},_{\xi}(\xi,0)$$

$$- (8/3)f_{22}s^2R\psi_{\eta},_{\eta}(\xi,0) - (8/3)f_{26}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0))$$

$$+ (32/9)h_{12}(s^2\psi_{\xi},_{\xi}(\xi,0)+s\bar{w},_{\xi\xi}(\xi,0)) + (32/9)h_{22}(s^2R\psi_{\eta},_{\eta}(\xi,0)+sR^2w,_{\eta\eta}(\xi,0))$$

$$+ (32/9)h_{26}(s^2R\psi_{\xi},_{\eta}(\xi,0)+s^2\psi_{\eta},_{\xi}(\xi,0)+2sRw,_{\xi\eta}(\xi,0))$$

$$- d_{12}s^2\psi_{\xi},_{\xi}(\xi,1) - d_{22}s^2R\psi_{\eta},_{\eta}(\xi,1) - d_{26}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1))$$

$$+ (4/3)f_{12}(s^2\psi_{\xi},_{\xi}(\xi,1)+s\bar{w},_{\xi\xi}(\xi,1)) + (4/3)f_{22}(s^2R\psi_{\eta},_{\eta}(\xi,1)+sR^2w,_{\eta\eta}(\xi,1))$$

$$+ (4/3)f_{26}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1)+2sR\bar{w},_{\xi\eta}(\xi,1)) + (8/3)f_{12}s^2\psi_{\xi},_{\xi}(\xi,1)$$

$$+ (8/3)f_{22}s^2R\psi_{\eta},_{\eta}(\xi,1) + (8/3)f_{26}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1))$$

$$- (32/9)h_{12}(s^2\psi_{\xi},_{\xi}(\xi,1)+s\bar{w},_{\xi\xi}(\xi,1)) - (32/9)h_{22}(s^2R\psi_{\eta},_{\eta}(\xi,1)+sR^2\bar{w},_{\eta\eta}(\xi,1))$$

$$- (32/9)h_{26}(s^2R\psi_{\xi},_{\eta}(\xi,1)+s^2\psi_{\eta},_{\xi}(\xi,1)+2sR\bar{w},_{\xi\eta}(\xi,1)) \Big\} \delta\psi_{\eta}\, d\xi = 0 \qquad (80)$$

Galerkin Equations

We apply the Galerkin technique by choosing three functions ψ_x, ψ_y, and w which have the following form [16]

$$X_{MN}(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn}\, F_{mn}(x,y) \qquad (81)$$

where c_{mn} are undetermined coefficients and the functions F_{mn} satisfy the geometric boundary conditions. By taking the first variation of Eq.(81), we obtain

$$\delta X_{1N} = \frac{\partial X_{11}}{\partial c_{11}}\,\delta c_{11} + \frac{\partial X_{12}}{\partial c_{12}}\,\delta c_{12} + \,\ldots\ldots = F_{11}\delta c_{11} + F_{12}\delta c_{12} + \,\ldots\ldots$$

$$\delta X_{2N} = \delta X_{1N} + \frac{\partial X_{21}}{\partial c_{21}}\,\delta c_{21} + \frac{\partial X_{22}}{\partial c_{22}}\,\delta c_{22} + \,\ldots = \delta X_{1N} + F_{21}\delta c_{21} + F_{22}\delta c_{22} + \,\ldots$$

$$\delta X_{3N} = \delta X_{1N} + \delta X_{2N} + \frac{\partial X_{31}}{\partial c_{31}}\,\delta c_{31} + \,\ldots = \delta X_{1N} + \delta X_{2N} + F_{31}\delta c_{31} + \,\ldots$$

$$\text{etc.} \tag{82}$$

or M equations with m x n terms.

The equations of motion are of the form

δw or Eq.(78) :

$$\int_0^1\int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta w \, d\xi d\eta + \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta w \, d\eta$$
$$+ \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta w \, d\xi + \{ f(\psi_\xi,\psi_\eta,w) \} \delta w = 0 \tag{83}$$

$\delta\psi_x$ or Eq.(79) :

$$\int_0^1\int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta\psi_x \, d\xi d\eta + \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta\psi_x \, d\eta$$
$$+ \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta\psi_x \, d\xi = 0 \tag{84}$$

$\delta\psi_y$ or Eq.(80) :

$$\int_0^1\int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta\psi_y \, d\xi d\eta + \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta\psi_y \, d\eta$$
$$+ \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} \delta\psi_y \, d\xi = 0 \tag{85}$$

where the integrands have been shown as functions f in terms of the three variables ψ_ξ, ψ_η, and w. Each integrand is multiplied by the first variation of the variable that the equation of motion describes. Using Eq.(81), we will write the admissible functions as

$$\psi_{x_{MN}}(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} F_{mn}(x,y)$$

$$\psi_{y_{MN}}(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn} F_{mn}(x,y) \tag{86}$$

$$w_{MN}(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} F_{mn}(x,y)$$

Using Eqs.(82), (84), and (86), we obtain

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \int_0^1\int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} F_{mn}\delta A_{mn}\, d\xi d\eta$$

$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} F_{mn}\delta A_{mn}\, d\eta$$

$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} F_{mn}\delta A_{mn}\, d\xi = 0 \tag{87}$$

In order for Eq.(87) to be identically zero, each individual term of each integral must be zero, since the constants A_{mn} are purely arbitrary. Because each term must vanish, the δA_{mn} constants can be taken outside of the integration and divided out. The following form is therefore obtained

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \int_0^1\int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} F_{mn}\, d\xi d\eta + \sum_{m=1}^{M} \sum_{n=1}^{N} \int_0^1 \{ f(\psi_\xi,\psi_\eta,w) \} F_{mn}\, d\eta$$

$$+ \sum_{m=1}^{M} \sum_{n=1}^{N} \int_0^1 \{ f(\psi_\xi, \psi_\eta, w) \} F_{mn} \, d\xi = 0 \tag{88}$$

Similar forms are obtained for the other equations of motion. The functions f contain the undetermined constants A_{mn}, B_{mn}, and C_{mn} whereas the functions F_{mn} do not. Since the approximate functions are double sums of m and n, and there are three independant variables w, ψ_x, and ψ_y, we now have three M by N matricies per equation of motion. This yields a 3 x M by 3 x N matrix which may be solved for the undetermined constants. Thus

$$\begin{array}{cc} \psi_x & : \\ \psi_y & : \\ w & : \end{array} \begin{Bmatrix} [A1] & [B1] & [C1] \\ [A2] & [B2] & [C2] \\ [A3] & [B3] & [C3] \end{Bmatrix} \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{Bmatrix} = 0 \tag{89}$$

Once the undetermined constants are found, they may be substituted back into the equations of motion to give an approximate solution. However, for this work we are only interested in natural frequencies and buckling loads. These solutions involve the formulation of an eigenvalue problem of the following form

$$
\begin{array}{ll}
\psi_x : \\
\psi_y : \\
w :
\end{array}
\begin{Bmatrix}
[A1] & [B1] & [C1] \\
[A2] & [B2] & [C2] \\
[A3] & [B3] & [C3]
\end{Bmatrix}
\begin{Bmatrix}
A_{mn} \\
B_{mn} \\
C_{mn}
\end{Bmatrix}
= \lambda_i
\begin{Bmatrix}
[X1] & [Y1] & [Z1] \\
[X2] & [Y2] & [Z2] \\
[X3] & [Y3] & [Z3]
\end{Bmatrix}
\begin{Bmatrix}
A_{mn} \\
B_{mn} \\
C_{mn}
\end{Bmatrix}
\qquad (90)
$$

where A, B, and C represent the stiffness terms and X, Y, and Z represent the mass/inertia terms. The mass/inertia terms are separated out from the stiffness terms so that it may be solved using existing computer routines.

Simply Supported Boundary Condition A plate which is simply supported on all four sides has the following requirements

@ x = 0, a :

$$w = 0 = \psi_y$$

$$M_x = D_{11}\psi_{x,x} + D_{12}\psi_{y,y} + D_{16}(\psi_{x,y}+\psi_{y,x}) - (4/3h^2)F_{11}(\psi_{x,x}+w_{,xx})$$
$$- (4/3h^2)F_{12}(\psi_{y,y}+w_{,yy}) - (4/3h^2)F_{16}(\psi_{x,y}+\psi_{y,x}+2w_{,xy}) = 0 \qquad (91)$$

and

@ y = 0, b :

$$w = 0 = \psi_x$$

$$M_y = D_{12}\psi_{x,x} + D_{22}\psi_{y,y} + D_{26}(\psi_{x,y}+\psi_{y,x}) - (4/3h^2)F_{12}(\psi_{x,x}+w_{,xx})$$
$$- (4/3h^2)F_{22}(\psi_{y,y}+w_{,yy}) - (4/3h^2)F_{26}(\psi_{x,y}+\psi_{y,x}+2w_{,xy}) = 0 \qquad (92)$$

Functions which satisfy the above geometric requirements (i.e., $w = \psi_x = 0$ @ x = 0, a) are referred to as admissible functions. The following set of functions are indeed admissible

$$\psi_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos(m\pi x/a) \sin(n\pi y/b)$$
$$\psi_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \qquad (93)$$
$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

Other sets of functions could be used in place of Eqs.(93). The above set was chosen because trigonometric functions are natural solutions to harmonic problems (such as vibrations and buckling) and are easy to manipulate mathematically. Normalizing Eqs.(93) we obtain

$$\psi_{\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi\xi) \cos(n\pi\eta) \tag{94}$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi\xi) \sin(n\pi\eta)$$

We next calculate the needed derivatives for substitution into the normalized equations of motion

$$\psi_{\xi',\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, m\pi \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, m^2\pi^2 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, m^3\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, mn\pi^2 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, m^2n\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, mn^2\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, n\pi \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, n^2\pi^2 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, n^3\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, n\pi \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, n^2\pi^2 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\, n^3\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, mn\pi^2 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta,\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\ m^2 n\pi^3 \sin(m\pi\xi)\sin(n\pi\eta) \tag{95}$$

$$\psi_{\eta,\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$\psi_{\eta,\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\ m\pi \cos(m\pi\xi)\cos(n\pi\eta)$$

$$\psi_{\eta,\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$\psi_{\eta,\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ m^3\pi^3 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ m\pi \cos(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ m^4\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ n\pi \sin(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ n^3\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\eta\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ n^4\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^3 n\pi^4 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\xi\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ m^2 n^2\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ mn^3\pi^4 \cos(m\pi\xi)\cos(n\pi\eta)$$

For the δw equation of motion we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1\int_0^1 \Big\{ (a_{45}-8d_{45}+16f_{45})(s^3B_{mn}\ m\pi \cos(m\pi\xi) \cos(n\pi\eta)$$

$$+s^3RA_{mn}\ n\pi \cos(m\pi\xi) \cos(n\pi\eta)+2s^2RC_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta))$$

$$+ (a_{55}-8d_{55}+16f_{55})(-s^3A_{mn}\ m\pi \sin(m\pi\xi) \sin(n\pi\eta)-s^2C_{mn}\ m^2\pi^2$$

$$\sin(m\pi\xi) \sin(n\pi\eta)) + (a_{44}-8d_{44}+16f_{44})(-s^3RB_{mn}\ n\pi \sin(m\pi\xi) \sin(n\pi\eta)$$

$$-s^2R^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)) + N_0k_1C_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$+ N_0R^2k_2C_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta) + N_0Rk_3C_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta))$$

$$+ (4/3)f_{11}sA_{mn}\ m^3\pi^3 \sin(m\pi\xi) \sin(n\pi\eta) + (4/3)sRf_{12}$$

$$B_{mn}\ m^2n\pi^3 \sin(m\pi\xi) \sin(n\pi\eta) + (4/3)f_{16}(-sRA_{mn}\ m^2n\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$-sB_{mn}\ m^3\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)) - (16/9)h_{11}(sA_{mn}\ m^3\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$+C_{mn}\ m^4\pi^4 \sin(m\pi\xi) \sin(n\pi\eta)) - (16/9)h_{12}(sRB_{mn}\ m^2n\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$+R^2C_{mn}\ m^2n^2\pi^4 \sin(m\pi\xi) \sin(n\pi\eta))- (16/9)h_{16}(-sRA_{mn}\ m^2n\pi^3$$

$$\cos(m\pi\xi) \cos(n\pi\eta)-sB_{mn}\ m^3\pi^3 \cos(m\pi\xi) \cos(n\pi\eta) - 2RC_{mn}\ m^3n\pi^4$$

$$\cos(m\pi\xi) \cos(n\pi\eta)) + (4/3)sR^2f_{12}A_{mn}\ mn^2\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$+ (4/3)sR^3f_{22}B_{mn}\ n^3\pi^3 \sin(m\pi\xi) \sin(n\pi\eta) + (4/3)f_{26}(-sR^3A_{mn}\ n^3\pi^3$$

$$\cos(m\pi\xi) \cos(n\pi\eta)-sR^2B_{mn}\ mn^2\pi^3 \cos(m\pi\xi) \cos(n\pi\eta))$$

$$- (16/9)h_{12}(sR^2A_{mn}\ mn^2\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)+R^2C_{mn}\ m^2n^2\pi^4$$

$$\sin(m\pi\xi) \sin(n\pi\eta)) - (16/9)h_{22}(sR^3B_{mn}\ n^3\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$+R^4C_{mn}\ n^4\pi^4 \sin(m\pi\xi) \sin(n\pi\eta)) - (16/9)h_{26}(-sR^3A_{mn}\ n^3\pi^3$$

$$\cos(m\pi\xi) \cos(n\pi\eta)-sR^2B_{mn}\ mn^2\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)-2R^3C_{mn}\ mn^3\pi^4$$

$$\cos(m\pi\xi) \cos(n\pi\eta)) - (8/3)sRf_{16}A_{mn}\ m^2n\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$- (8/3)sR^2f_{26}B_{mn}\ mn^2\pi^3 \cos(m\pi\xi) \cos(n\pi\eta) + (8/3)f_{66}(sR^2A_{mn}\ mn^2\pi^3$$

$$\sin(m\pi\xi) \sin(n\pi\eta)+sRB_{mn}\ m^2n\pi^3 \sin(m\pi\xi) \sin(n\pi\eta))$$

$$- (32/9)h_{16}(-sRA_{mn}\ m^2n\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)-RC_{mn}\ m^3n\pi^4 \cos(m\pi\xi) \cos(n\pi\eta))$$

$$
\begin{aligned}
& - (32/9)h_{26}(-sR^2B_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\cos(n\pi\eta) - R^3C_{mn}\ mn^3\pi^4 \\
& \cos(m\pi\xi)\cos(n\pi\eta)) - (32/9)h_{66}(sR^2A_{mn}\ mn^2\pi^3 \sin(m\pi\xi)\sin(n\pi\eta) \\
& + sRB_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(n\pi\eta) + 2R^2C_{mn}\ m^2n^2\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)) \\
& + \omega^2\ (\ s^2C_{mn} \sin(m\pi\xi)\sin(n\pi\eta) + 16/4032C_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) \\
& + 16/4032R^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) - 4/315sA_{mn}\ m\pi \sin(m\pi\xi)\sin(n\pi\eta) \\
& -4/315sRB_{mn}\ n\pi \sin(m\pi\xi)\sin(n\pi\eta)\ )\Bigg\}\Bigg\{\sin(p\pi\xi)\sin(q\pi\eta)\Bigg\}\, d\xi d\eta = 0
\end{aligned}
\tag{96}
$$

where by inspection, the boundary terms are zero.

We now multiply through by $\sin(p\pi\xi)\sin(q\pi\eta)$

$$
\begin{aligned}
& \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1\!\!\int_0^1 \Bigg\{ (\ a_{45}-8d_{45}+16f_{45}\ )(s^3B_{mn}\ m\pi \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta) \\
& \sin(qn\pi) + s^3RA_{mn}\ n\pi \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) + 2s^2RC_{mn}\ mn\pi^2 \\
& \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) + (a_{55}-8d_{55}+16f_{55}\ ) \\
& (-s^3A_{mn}\ m\pi \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) - s^2C_{mn}\ m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi) \\
& \sin(n\pi\eta)\sin(q\pi\eta)) + (\ a_{44}-8d_{44}+16f_{44}\ )(-s^3RB_{mn}\ n\pi \sin(m\pi\xi)\sin(p\pi\xi) \\
& \sin(n\pi\eta)\sin(q\pi\eta) - s^2R^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)) \\
& + N_0k_1C_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + N_0R^2k_2C_{mn}\ n^2\pi^2 \\
& \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + N_0Rk_3C_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \\
& \cos(n\pi\eta)\sin(q\pi\eta)\ ) + (4/3)f_{11}sA_{mn}\ m^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \\
& \sin(n\pi\eta)\sin(q\pi\eta) + (4/3)sRf_{12}B_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \\
& + (4/3)f_{16}(-sRA_{mn}\ m^2n\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) \\
& -sB_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{11}(sA_{mn}\ m^3\pi^3 \\
& \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + C_{mn}\ m^4\pi^4 \sin(m\pi\xi)\sin(p\pi\xi) \\
& \sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{12}(sRB_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \\
& \sin(n\pi\eta)\sin(q\pi\eta) + R^2C_{mn}\ m^2n^2\pi^4 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta))
\end{aligned}
$$

$$
\begin{aligned}
&- (16/9)h_{16}(-sRA_{mn}\, m^2n\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) \\
&-sB_{mn}\, m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - 2RC_{mn}\, m^3n\pi^4 \\
&\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) + (4/3)sR^2f_{12}A_{mn}\, mn^2\pi^3 \\
&\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + (4/3)sR^3f_{22}B_{mn}\, n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \\
&\sin(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{26}(-sR^3A_{mn}\, n^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \\
&\cos(n\pi\eta)\sin(q\pi\eta) - sR^2B_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) \\
&- (16/9)h_{12}(sR^2A_{mn}\, mn^2\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + R^2C_{mn}\, m^2n^2\pi^4 \\
&\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{22}(sR^3B_{mn}\, n^3\pi^3 \\
&\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + R^4C_{mn}\, n^4\pi^4 \sin(m\pi\xi)\sin(p\pi\xi) \\
&\sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{26}(-sR^3A_{mn}\, n^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \\
&\cos(n\pi\eta)\sin(q\pi\eta) - sR^2B_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) \\
&-2R^3C_{mn}\, mn^3\pi^4 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) - (8/3)sRf_{16}A_{mn}\, m^2n\pi^3 \\
&\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - (8/3)sR^2f_{26}B_{mn}\, mn^2\pi^3 \\
&\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) + (8/3)f_{66}(sR^2A_{mn}\, mn^2\pi^3 \\
&\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + sRB_{mn}\, m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \\
&\sin(n\pi\eta)\sin(q\pi\eta)) - (32/9)h_{16}(-sRA_{mn}\, m^2n\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \\
&\cos(n\pi\eta)\sin(q\pi\eta) - RC_{mn}\, m^3n\pi^4 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) \\
&- (32/9)h_{26}(-sR^2B_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - R^3C_{mn}\, mn^3\pi^4 \\
&\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) - (32/9)h_{66}(sR^2A_{mn}\, mn^2\pi^3 \\
&\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + sRB_{mn}\, m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \\
&\sin(n\pi\eta)\sin(q\pi\eta) + 2R^2C_{mn}\, m^2n^2\pi^4 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \\
&+ \omega^2 \,(\, s^2C_{mn} \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + 16/4032C_{mn}\, m^2\pi^2 \\
&\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + 16/4032R^2C_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \\
&\sin(n\pi\eta)\sin(q\pi\eta) - 4/315sA_{mn}\, m\pi \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) \\
&-4/315sRB_{mn}\, n\pi \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)\,)\, \Big\}\, d\xi d\eta = 0 \qquad (97)
\end{aligned}
$$

We now simplify the integration of the previous equation. The following definite integrals will be used

For $m = p$ or $n = q$:

$$\cos(m\pi x)\cos(n\pi x)\ dx = 1/2$$

$$\sin(m\pi x)\sin(p\pi x)\ dx = 1/2 \tag{98}$$

$$\sin(m\pi x)\cos(p\pi x)\ dx = 0$$

For $m \neq p$ or $n \neq q$:

$$\cos(m\pi x)\cos(p\pi x)\ dx = 0$$

$$\sin(m\pi x)\sin(p\pi x)\ dx = 0 \tag{99}$$

$$\sin(m\pi x)\cos(p\pi x)\ dx \begin{cases} = 0 \text{ for } (m+p) \text{ even integer} \\ =2m/\pi(m^2-p^2) \text{ for } (m+p) \text{ odd integer} \end{cases}$$

$$\cos(m\pi x)\sin(p\pi x)\ dx \begin{cases} = 0 \text{ for } (m+p) \text{ even integer} \\ =2p/\pi(p^2-m^2) \text{ for } (m+p) \text{ odd integer} \end{cases}$$

The notation to be used is

$$\cos(m\pi x)\cos(p\pi x)\ dx = (\ 1/2 \text{ or } 0\ ) \tag{100}$$

where the left side indicates the condition of $m=p$ and the right side indicates the condition $m\neq p$. Thus

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\Big[(1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^3 m\pi)(-a_{55}+8d_{55}-16f_{55}) + (1/2\text{or}0)(1/2\text{or}0)$$
$$(sm^3\pi^3)(4/3f_{11}-16/9h_{11}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(sR^2mn^2\pi^3)(4/3f_{12}-16/9h_{12}$$
$$+8/3f_{66}-32/9h_{66}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(-4/315\bar{\omega}^2 sm\pi) + (0 \text{ or } 0 \text{ even},$$
$$2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even}, 2q/\pi(q^2-n^2) \text{ odd})(s^3Rn\pi)(a_{45}-8d_{45}+16f_{45})$$

$$(0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even}, 2q/\pi(q^2-n^2) \text{ odd})(sRm^2n\pi^2)$$

$$(-4f_{16}+48/9h_{16}) + (0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},$$

$$\left. 2q/\pi(q^2-n^2) \text{ odd})(sR^3n^3\pi^3)(-4/3f_{26}+16/9h_{26}) \right] A_{mn}$$

$$+ \Big[(1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^3Rn\pi)(-a_{44}+8d_{44}-16f_{44}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)$$

$$(sRm^2n\pi^3)(4/3f_{12}-16/9h_{12}+8/3f_{66}-32/9h_{26}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)$$

$$(sR^3n^3\pi^3)(4/3f_{22}-16/9h_{22}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(-4/315\bar{\omega}^2sRn\pi)$$

$$(0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even}, 2q/\pi(q^2-n^2) \text{ odd})(sm\pi)$$

$$(a_{45}-8d_{45}+16f_{45}) + (0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},$$

$$2q/\pi(q^2-n^2) \text{ odd})(sm^3\pi^3)(16/9h_{16}) + (0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})$$

$$\left. (0 \text{ or } 0 \text{ even}, 2q/\pi(q^2-n^2) \text{ odd})(sR^2mn^2\pi^3)(-4f_{26}+48/9h_{26}) \right] B_{mn}$$

$$+ \Big[(1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2m^2\pi^2)(-a_{55}+8d_{55}-16f_{55}) + (1/2 \text{ or } 0)$$

$$(1/2 \text{ or } 0)(s^2R^2n^2\pi^2)(-a_{44}+8d_{44}-16f_{44}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(\bar{N}_0k_1m^2\pi^2$$

$$+\bar{N}_0R^2k_2n^2\pi^2+\bar{\omega}^2s^2+16/4032\bar{\omega}^2m^2\pi^2+16/4032\bar{\omega}^2R^2n^2\pi^2) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)$$

$$(\pi^4)(-16/9h_{11}m^4-32/9h_{12}R^2m^2n^2-16/9h_{22}R^4n^4-64/9h_{66}R^2m^2n^2)$$

$$(0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even}, 2q/\pi(q^2-n^2) \text{ odd})(2s^2Rmn\pi^2)$$

$$(a_{45}-8d_{45}+16f_{45})(0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},$$

$$2q/\pi(q^2-n^2) \text{ odd})(\bar{N}_0Rk_3mn\pi^2) + (0 \text{ or } 0 \text{ even}, 2p/\pi(p^2-m^2) \text{ odd})$$

$$\left. \left. (0 \text{ or } 0 \text{ even}, 2q/\pi(q^2-n^2) \text{ odd})(64/9\pi^4)(h_{16}Rm^3n+h_{26}R^3mn^3) \right] C_{mn} \right\} = 0 \tag{101}$$

These equations will be used in a computer program where

$$LAM1 = a^2p1/\ E_2h = \bar{\omega}^2/\ \omega^2 \tag{102}$$

ω^2 is factored out of the mass/inertia matrix and is found by solution

of the eigenvalue problem (see Eq.(90) on page 61). We will also use

$$LAM2 = a^2/ E_2h^3 = \bar{N}_0/ N_0 \tag{103}$$

N_0 is factored out of the mass/inertia and is also found using the eigenvalue routine. The same terms will be used in the other equations of motion.

The integration functions to be used in the computer program are shown below.

For m and p:

$$\begin{gathered} F1 = (1/2 \text{ or } 0) \\ F3 = (0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd}) \\ F5 = (0 \text{ or } 0 \text{ even},\ 2m/\pi(m^2-p^2) \text{ odd}) \end{gathered} \tag{104}$$

For n and q:

$$\begin{gathered} F2 = (1/2 \text{ or } 0) \\ F4 = (0 \text{ or } 0 \text{ even},\ 2q/\pi(q^2-n^2) \text{ odd}) \\ F6 = (0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd}) \end{gathered} \tag{105}$$

For the $\delta\psi_x$ equation of motion we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\int_0^1 \Big\{ -d_{11}s^2A_{mn}\ m^2\pi^2 \cos(m\pi\xi) \sin(n\pi\eta) - d_{12}s^2RB_{mn}\ mn\pi^2$$
$$\cos(m\pi\xi) \sin(n\pi\eta) + d_{16}(-s^2RA_{mn}\ mn\pi^2 \sin(m\pi\xi) \cos(n\pi\eta)$$
$$-s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi) \cos(n\pi\eta)) - (4/3)f_{11}(-s^2A_{mn}\ m^2\pi^2 \cos(m\pi\xi) \sin(n\pi\eta)$$
$$-sC_{mn}\ m^3\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)) - (4/3)f_{12}(-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi) \sin(n\pi\eta)$$
$$-sR^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)) - (4/3)f_{16}(-s^2RA_{mn}\ mn\pi^2$$
$$\sin(m\pi\xi) \cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi) \cos(n\pi\eta)-2sRC_{mn}\ m^2n\pi^3$$

$$\sin(m\pi\xi)\cos(n\pi\eta)) - d_{16}s^2RA_{mn}\; mn\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$- d_{26}s^2R^2B_{mn}\; n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) + d_{66}(-s^2R^2A_{mn}\; n^2\pi^2$$

$$\cos(m\pi\xi)\sin(n\pi\eta) - s^2RB_{mn}\; mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)) - (4/3)f_{16}(-s^2RA_{mn}$$

$$mn\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - sRC_{mn}\; m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta))$$

$$- (4/3)f_{26}(-s^2R^2B_{mn}\; n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - sR^3C_{mn}\; n^3\pi^3$$

$$\sin(m\pi\xi)\cos(n\pi\eta)) - (4/3)f_{66}(-s^2R^2A_{mn}\; n^2\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$-s^2RB_{mn}\; mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) - 2sR^2C_{mn}\; mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta))$$

$$- (\; a_{45} - 8d_{45} + 16f_{45}\;)(s^4B_{mn} \sin(m\pi\xi)\cos(n\pi\eta) + s^3RC_{mn}\; n\pi$$

$$\sin(m\pi\xi)\cos(n\pi\eta)) - (\; a_{55} - 8d_{55} + 16f_{55}\;)(s^4A_{mn} \cos(m\pi\xi)\sin(n\pi\eta)$$

$$+s^3C_{mn}\; m\pi \cos(m\pi\xi)\sin(n\pi\eta)) + (4/3)f_{11}s^2A_{mn}\; m^2\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$+ (4/3)\; f_{12}s^2RB_{mn}\; mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) - (4/3)f_{16}(-s^2RA_{mn}\; mn\pi^2$$

$$\sin(m\pi\xi)\cos(n\pi\eta) - s^2B_{mn}\; m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)) + (16/9)h_{11}$$

$$(-s^2A_{mn}\; m^2\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) - sC_{mn}\; m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta))$$

$$+ (16/9)h_{12}(-s^2RB_{mn}\; mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) - sR^2C_{mn}\; mn^2\pi^3$$

$$\cos(m\pi\xi)\sin(n\pi\eta)) + (16/9)h_{16}(-s^2RA_{mn}\; mn\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$-s^2B_{mn}\; m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - 2sRC_{mn}\; m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta))$$

$$+ (4/3)f_{16}s^2RA_{mn}\; mn\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) + (4/3)f_{26}s^2R^2B_{mn}\; n^2\pi^2$$

$$\sin(m\pi\xi)\cos(n\pi\eta) + (4/3)f_{66}(s^2R^2A_{mn}\; n^2\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$-s^2RB_{mn}\; mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)) - (16/9)h_{16}(s^2RA_{mn}\; mn\pi^2$$

$$\sin(m\pi\xi)\cos(n\pi\eta) - sRC_{mn}\; m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)) + (16/9)h_{26}(-s^2R^2B_{mn}$$

$$n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - sR^3C_{mn}\; n^3\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)) + (16/9)h_{66}$$

$$(-s^2R^2A_{mn}\; n^2\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) - s^2RB_{mn}\; mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$-2sR^2C_{mn}\; mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) + 17/315\omega^2s^2A_{mn} \cos(m\pi\xi)\sin(n\pi\eta)$$

$$- 4/315\bar{\omega}^2sC_{mn}\; m\pi \cos(m\pi\xi)\sin(n\pi\eta) \Big\} \Big\{ \cos(p\pi\xi)\sin(q\pi\eta) \Big\}\, d\xi d\eta$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \Big\{ \Big[-d_{11}s^2A_{mn}\; m\pi \sin(0)\sin(n\pi\eta) - d_{12}s^2RB_{mn}\; n\pi \sin(0)$$

$$\sin(n\pi\eta) + d_{16}(s^2RA_{mn}\ n\pi\ \cos(0)\ \cos(n\pi\eta)+s^2B_{mn}\ m\pi\ \cos(0)\ \cos(n\pi\eta))$$

$$-\ (4/3)f_{11}(-s^2A_{mn}\ m\pi\ \sin(0)\ \sin(n\pi\eta)-sC_{mn}\ m^2\pi^2\ \sin(0)\ \sin(n\pi\eta))$$

$$-\ (4/3)f_{12}(-s^2RB_{mn}\ n\pi\ \sin(0)\ \sin(n\pi\eta)-sR^2C_{mn}\ n^2\pi^2\ \sin(0)\ \sin(n\pi\eta))$$

$$-\ (4/3)f_{16}(s^2RA_{mn}\ n\pi\ \cos(0)\ \cos(n\pi\eta)+s^2B_{mn}\ m\pi\ \cos(0)\ \cos(n\pi\eta)$$

$$+2sRC_{mn}\ mn\pi^2\ \cos(0)\ \cos(n\pi\eta)) + (8/3)f_{11}s^2A_{mn}\ m\pi\ \sin(0)\ \sin(n\pi\eta)$$

$$+\ (8/3)f_{12}s^2RB_{mn}\ n\pi\ \sin(0)\ \sin(n\pi\eta) - (8/3)f_{16}(s^2RA_{mn}\ n\pi\ \cos(0)\ \cos(n\pi\eta)$$

$$+s^2B_{mn}\ m\pi\ \cos(0)\ \cos(n\pi\eta)) + (32/9)h_{11}(-s^2A_{mn}\ m\pi\ \sin(0)\ \sin(n\pi\eta)$$

$$-sC_{mn}\ m^2\pi^2\ \sin(0)\ \sin(n\pi\eta)) + (32/9)h_{12}(-s^2RB_{mn}\ n\pi\ \sin(0)\ \sin(n\pi\eta)$$

$$-sR^2C_{mn}\ n^2\pi^2\ \sin(0)\ \sin(n\pi\eta)) + (32/9)h_{16}(s^2RA_{mn}\ n\pi\ \cos(0)\ \cos(n\pi\eta)$$

$$+s^2B_{mn}\ m\pi\ \cos(0)\ \cos(n\pi\eta)+2s^2RC_{mn}\ mn\pi^2\ \cos(0)\ \cos(n\pi\eta))\ \Big]$$

$$\Big[\ \cos(0)\ \sin(q\pi\eta)\ \Big]$$

$$+\ \Big[\ d_{11}s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta) + d_{12}s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$-\ d_{16}(s^2RA_{mn}\ n\pi\ \cos(m\pi)\ \cos(n\pi\eta)+s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta))$$

$$-\ (4/3)f_{11}(s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta)+sC_{mn}\ m^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta))$$

$$-\ (4/3)f_{12}(s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta)+sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta))$$

$$-\ (4/3)f_{16}(-s^2RA_{mn}\ n\pi\ \cos(m\pi)\ \cos(n\pi\eta)-s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta)$$

$$-2sRC_{mn}\ mn\pi^2\ \cos(m\pi)\ \cos(n\pi\eta)) - (8/3)f_{11}s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$-\ (8/3)f_{12}s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta) - (8/3)f_{16}(-s^2RA_{mn}\ n\pi\ \cos(m\pi)$$

$$\cos(n\pi\eta)-s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta)) + (32/9)h_{11}(s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$+sC_{mn}\ m^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta)) + (32/9)h_{12}(s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$+sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta)) + (32/9)h_{16}(-s^2RA_{mn}\ n\pi\ \cos(m\pi)\ \cos(n\pi\eta)$$

$$-s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta)-2s^2RC_{mn}\ mn\pi^2\ \cos(m\pi)\ \cos(n\pi\eta))\ \Big]$$

$$\Big[\ \cos(p\pi)\ \sin(q\pi\eta)\ \Big]\ \Big\}\ d\eta$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1} \Bigg\{ \Bigg[-d_{16}s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(0) - d_{26}s^2rB_{mn}\ n\pi \sin(m\pi\xi)$$

$$\sin(0) + d_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(0)+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(0))$$

$$- (4/3)f_{16}(-s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(0)-sC_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(0))$$

$$- (4/3)f_{26}(-s^2RB_{mn}\ n\pi \sin(m\pi\xi) \sin(0)-sR^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(0))$$

$$- (4/3)f_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(0)+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(0)$$

$$+2sRC_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(0)) + (4/3)f_{16}s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(0)$$

$$+ (4/3)f_{26}s^2RB_{mn}\ n\pi \sin(m\pi\xi) \sin(0) - (4/3)f_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(0)$$

$$+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(0)) + (16/9)h_{16}(-s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(0)$$

$$-sC_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(0)) + (16/9)h_{26}(-s^2RB_{mn}\ n\pi \sin(m\pi\xi) \sin(0)$$

$$-sR^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(0))+ (16/9)h_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(0)$$

$$+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(0)+2sRC_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(0)) \Bigg]$$

$$\Big[\cos(p\pi\xi) \sin(0) \Big]$$

$$+ \Bigg[d_{16}s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(n\pi) + d_{26}s^2rB_{mn}\ n\pi \sin(m\pi\xi) \sin(n\pi)$$

$$- d_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(n\pi)+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(n\pi))$$

$$+ (4/3)f_{16}(-s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(n\pi)-sC_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(n\pi))$$

$$+ (4/3)f_{26}(-s^2RB_{mn}\ n\pi \sin(m\pi\xi) \sin(n\pi)-sR^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(n\pi))$$

$$+ (4/3)f_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(n\pi)+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(n\pi)$$

$$+2sRC_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(n\pi)) - (4/3)f_{16}s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(n\pi)$$

$$- (4/3)f_{26}s^2RB_{mn}\ n\pi \sin(m\pi\xi) \sin(n\pi)+(4/3)f_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(n\pi)$$

$$+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(n\pi)) - (16/9)h_{16}(-s^2A_{mn}\ m\pi \sin(m\pi\xi) \sin(n\pi)$$

$$-sC_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(n\pi)) - (16/9)h_{26}(-s^2RB_{mn}\ n\pi \sin(m\pi\xi) \sin(n\pi)$$

$$-sR^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(n\pi)) + (16/9)h_{66}(s^2RA_{mn}\ n\pi \cos(m\pi\xi) \cos(n\pi)$$

$$+s^2B_{mn}\ m\pi \cos(m\pi\xi) \cos(n\pi)+2sRC_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(n\pi)) \Bigg]$$

$$\left[\cos(p\pi\xi)\ \sin(q\pi) \right] \Bigg\} d\xi = 0 \qquad (106)$$

We now simplify the previous equation whereby $\sin(0) = \sin(m\pi) = 0$ and $\cos(0) = 1$. Thus

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\int_0^1 \Bigg\{ -d_{11}s^2A_{mn}\ m^2\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - d_{12}s^2 RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) + d_{16}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi) \cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (4/3)f_{11}(-s^2A_{mn}\ m^2\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - sC_{mn}\ m^3\pi^3 \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{12}(-s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi) \cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\cos(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) - s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - 2sRC_{mn} m^2n\pi^3\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) - d_{16}s^2RA_{mn}\ mn\pi^2 \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - d_{26}s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) + d_{66}(-s^2R^2A_{mn}\ n^2\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) -s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}(-s^2RA_{mn} mn\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\cos(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{26}(-s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) - sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (4/3)f_{66}(-s^2R^2A_{mn}\ n^2\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) -s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - 2sR^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\cos(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)) - (\ a_{45} - 8d_{45} + 16f_{45}\ )(s^4B_{mn}\ \sin(m\pi\xi) \cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) + s^3RC_{mn}\ n\pi\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (\ a_{55} - 8d_{55} + 16f_{55}\ )(s^4A_{mn}\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)$$

$$
\begin{aligned}
&+s^3C_{mn}\ m\pi\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)) + (4/3)f_{11}s^2A_{mn}\ m^2\pi^2 \\
&\cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) + (4/3)\ f_{12}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi) \\
&\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - (4/3)f_{16}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi) \\
&\cos(n\pi\eta)\sin(q\pi\eta) - s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) \\
&+ (16/9)h_{11}(-s^2A_{mn}\ m^2\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) \\
&-sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{12}(-s^2RB_{mn}\ mn\pi^2 \\
&\cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - sR^2C_{mn}\ mn^2\pi^3\cos(m\pi\xi)\cos(p\pi\xi) \\
&\sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{16}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta) \\
&\sin(q\pi\eta) - s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - 2sRC_{mn}\ m^2n\pi^3 \\
&\sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) + (4/3)f_{16}s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi) \\
&\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{26}s^2R^2B_{mn}\ n^2\pi^2\sin(m\pi\xi)\cos(p\pi\xi) \\
&\cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{66}(s^2R^2A_{mn}\ n^2\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta) \\
&\sin(q\pi\eta) - s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{16} \\
&(s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - sRC_{mn}\ m^2n\pi^3 \\
&\sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{26}(-s^2R^2B_{mn}n^2\pi^2\ \sin(m\pi\xi) \\
&\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\cos(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) \\
&+ (16/9)h_{66}(-s^2R^2A_{mn}\ n^2\pi^2\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - s^2RB_{mn}\ mn\pi^2 \\
&\cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - 2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\cos(p\pi\xi) \\
&\sin(n\pi\eta)\sin(q\pi\eta)) + 17/315\omega^2s^2A_{mn}\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) \\
&\left. - 4/315\bar{\omega}^2sC_{mn}\ m\pi\ \cos(m\pi\xi)\cos(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)\ \right\}\ d\xi\ d\eta \\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1 \Big\{\ d_{16}(s^2RA_{mn}\ n\pi\ \cos(n\pi\eta)\sin(q\pi\eta) + s^2B_{mn}\ m\pi\ \cos(n\pi\eta)\sin(q\pi\eta)) \\
&- (4/3)f_{16}(s^2RA_{mn}\ n\pi\ \cos(n\pi\eta)\sin(q\pi\eta) + s^2B_{mn}\ m\pi\ \cos(n\pi\eta)\sin(q\pi\eta) \\
&+2sRC_{mn}\ mn\pi^2\ \cos(n\pi\eta)\sin(q\pi\eta)) - (8/3)f_{16}(s^2RA_{mn}\ n\pi\ \cos(n\pi\eta)\sin(q\pi\eta) \\
&+s^2B_{mn}\ m\pi\ \cos(n\pi\eta)\sin(q\pi\eta)) + (32/9)h_{16}(s^2RA_{mn}\ n\pi\ \cos(n\pi\eta)\sin(q\pi\eta) \\
&+s^2B_{mn}\ m\pi\ \cos(n\pi\eta)\sin(q\pi\eta) + 2s^2RC_{mn}\ mn\pi^2\ \cos(n\pi\eta)\sin(q\pi\eta))\ \Big\}
\end{aligned}
$$

$- d_{16}(s^2RA_{mn}\ n\pi\ \cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)+s^2B_{mn}\ m\pi\ \cos(m\pi)\cos(p\pi)$
$\cos(n\pi\eta)\sin(q\pi\eta))+\ (4/3)f_{16}(s^2RA_{mn}\ n\pi\ \cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)$
$+s^2B_{mn}\ m\pi\ \cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)+2sRC_{mn}\ mn\pi^2\ \cos(m\pi)\cos(p\pi)$
$\cos(n\pi\eta)\sin(q\pi\eta))\ +\ (8/3)f_{16}(s^2RA_{mn}\ n\pi\ \cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)$
$+s^2B_{mn}\ m\pi\ \cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta))\ -\ (32/9)h_{16}(s^2RA_{mn}\ n\pi$
$\cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)+s^2B_{mn}\ m\pi\ \cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta)$
$$+2s^2RC_{mn}\ mn\pi^2\ \cos(m\pi)\cos(p\pi)\cos(n\pi\eta)\sin(q\pi\eta))\ \Big\}\ d\eta = 0 \qquad (107)$$

Using the integration notation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg\{\Bigg[\ (1/2\text{ or }0)(1/2\text{ or }0)(s^2m^2\pi^2)(-d_{11}+8/3f_{11}-16/9h_{11})\ +\ (1/2\text{or}0)$$
$(1/2\text{ or }0)(s^2R^2n^2\pi^2)(4/3f_{16}-d_{66}+8/3f_{66}-16/9h_{66})\ +\ (1/2\text{ or }0)(1/2\text{ or }0)$
$(s^4)(-a_{55}+8d_{55}-16f_{55})\ +\ (1/2\text{ or }0)(1/2\text{ or }0)(17/315\bar{\omega}^2s^2)$
$(0\text{ or }0\text{ even, }2m/\pi(m^2-p^2)\text{ odd})(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(s^2Rmn\pi^2)$
$(-2d_{16}+4/3f_{12}+16/3f_{16}-32/9h_{16})\ +\ (0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})$
$(1-\cos(m\pi)\cos(p\pi))(s^2Rn\pi)(d_{16}-4f_{16}+32/9h_{16})\ \Bigg]\ A_{mn}$
$+\ \Bigg[\ (1/2\text{or}0)(1/2\text{or}0)(s^2Rmn\pi^2)(-d_{12}-d_{66}+8/3f_{66}+4/3f_{12}-16/9h_{12}-16/9h_{66})$
$(0\text{ or }0\text{ even, }2m/\pi(m^2-p^2)\text{ odd})(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(s^2m^2\pi^2)$
$(-d_{16}+4/3f_{16}-16/9h_{16})\ +\ (0\text{ or }0\text{ even, }2m/\pi(m^2-p^2)\text{ odd})$
$(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})\ (s^4)\ (-a_{45}+8d_{45}-16f_{45})\ +$
$(0\text{ or }0\text{ even, }2m/\pi(m^2-p^2)\text{ odd})(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(s^2R^2n^2\pi^2)$
$(-d_{26}+8/3f_{26}-16/9h_{26})(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(1-\cos(m\pi)\cos(p\pi))$
$(s^2m\pi)(d_{16}-4f_{16}+32/9h_{16})\ \Bigg]\ B_{mn}$
$+\ \Bigg[\ (1/2\text{ or }0)(1/2\text{ or }0)(sR^2mn^2\pi^3)(4/3f_{12}+8/3f_{66}-16/9h_{12}-32/9h_{66})$

$(1/2 \text{ or } 0)(1/2 \text{ or } 0)(sm^3\pi^3)(-16/9h_{11}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(sm^3\pi)$

$(-a_{55}+8d_{55}-16f_{55}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(-4/315\bar{\omega}^{2}sm\pi)$

$(0 \text{ or } 0 \text{ even, } 2m/\pi(m^2-p^2) \text{ odd})(0 \text{ or } 0 \text{ even, } 2q/\pi(q^2-n^2) \text{ odd})(sRm^2n\pi^3)$

$(4f_{16}-48/9h_{16}) + (0 \text{ or } 0 \text{ even, } 2m/\pi(m^2-p^2) \text{ odd})$

$(0 \text{ or } 0 \text{ even, } 2m/\pi(m^2-p^2) \text{ odd})(sR^3n^3\pi^3)(4/3f_{26}-16/9h_{26})$

$(0 \text{ or } 0 \text{ even, } 2q/\pi(q^2-n^2) \text{ odd})(1-\cos(m\pi)\cos(p\pi))(sRmn\pi^2)$

$$(-8/3f_{16}+64/9h_{16}) \Big] C_{mn} \Big\} = 0 \qquad (108)$$

For the $\delta\psi_y$ equation of motion we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\int_0^1 \Big\{ -d_{12}s^2RA_{mn}\, mn\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - d_{22}s^2R^2B_{mn}\, n^2\pi^2$$

$\sin(m\pi\xi)\cos(n\pi\eta) + d_{26}(-s^2R^2A_{mn}\, n^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta)-s^2RB_{mn}\, mn\pi^2$

$\cos(m\pi\xi)\sin(n\pi\eta)) - (4/3)f_{12}(-s^2RA_{mn}\, mn\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$

$-sRC_{mn}\, m^2n\pi^3\sin(m\pi\xi)\cos(n\pi\eta)) - (4/3)f_{22}(-s^2R^2B_{mn}\, n^2\pi^2\sin(m\pi\xi)$

$\cos(n\pi\eta)-sR^3C_{mn}\, n^3\pi^3\sin(m\pi\xi)\cos(n\pi\eta))$

$- (4/3)f_{26}(-s^2R^2A_{mn}\, n^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta)-s^2RB_{mn}\, mn\pi^2\cos(m\pi\xi)\sin(n\pi\eta)$

$-2sR^2C_{mn}\, mn^2\pi^3\cos(m\pi\xi)\sin(n\pi\eta)) - d_{16}s^2A_{mn}\, m^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta)$

$- d_{26}s^2RB_{mn}\, mn\pi^2\cos(m\pi\xi)\sin(n\pi\eta) + d_{66}(-s^2RA_{mn}\, mn\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$

$-sB_{mn}\, m^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)) - (4/3)f_{16}(-s^2A_{mn}\, m^2\pi^2\cos(m\pi\xi)\sin(n\pi\eta)$

$-sC_{mn}\, m^3\pi^3\cos(m\pi\xi)\sin(n\pi\eta)) - (4/3)f_{26}(-s^2RA_{mn}\, mn\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$

$-s^2R^2C_{mn}\, mn^2\pi^3\cos(m\pi\xi)\sin(n\pi\eta) - (4/3)f_{66}(-s^2RA_{mn}\, mn\pi^2\sin(m\pi\xi)$

$\cos(n\pi\eta)-s^2B_{mn}\, m^2\pi^2\sin(m\pi\xi)\cos(n\pi\eta)-2s^2RC_{mn}\, m^2n\pi^3\sin(m\pi\xi)\cos(n\pi\eta))$

$- (a_{44}-8d_{44}+16f_{44})(s^4B_{mn}\sin(m\pi\xi)\cos(n\pi\eta)+s^3RC_{mn}\, n\pi\sin(m\pi\xi)$

$\cos(n\pi\eta)) - (a_{45}-8d_{45}+16f_{45})(s^4A_{mn}\cos(m\pi\xi)\sin(n\pi\eta)+s^3C_{mn}\, m\pi$

$\cos(m\pi\xi)\sin(n\pi\eta)) + (4/3)f_{12}s^2RA_{mn}\, mn\pi^2\sin(m\pi\xi)\cos(n\pi\eta)$

$$+ (4/3)f_{22}s^2R^2B_{mn}\ n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - (4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2$$

$$\cos(m\pi\xi)\sin(n\pi\eta) - s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta))$$

$$+ (16/9)h_{12}(-s^2RA_{mn}\ mn\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - sRC_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta))$$

$$+ (16/9)h_{22}(-s^2R^2B_{mn}\ n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - sR^3C_{mn}\ n^3\pi^3 \sin(m\pi\xi)\cos(n\pi\eta))$$

$$+ (16/9)h_{26}(-s^2R^2A_{mn}\ n^2\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) - s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$-2sR^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) + (4/3)f_{16}s^2A_{mn}\ m^2\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$+ (4/3)f_{26}s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) + (4/3)f_{66}(s^2RA_{mn}\ mn\pi^2 \sin(m\pi\xi)$$

$$\cos(n\pi\eta) - s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)) + (16/9)h_{16}(-s^2A_{mn}\ m^2\pi^2$$

$$\cos(m\pi\xi)\sin(n\pi\eta) - sC_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) + (16/9)h_{26}(-s^2RB_{mn}\ mn\pi^2$$

$$\cos(m\pi\xi)\sin(n\pi\eta) - s^2R^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta))$$

$$+ (16/9)h_{66}(-s^2RA_{mn}\ mn\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) - s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$-2sRC_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)) + \omega^2 \Big(17/315s^2B_{mn} \sin(m\pi\xi)$$

$$\cos(n\pi\eta) - 4/315sC_{mn}\ n\pi \sin(m\pi\xi)\cos(n\pi\eta)\Big) \Bigg\}$$

$$\Bigg\{ \sin(p\pi\xi)\cos(q\pi\eta) \Bigg\}\, d\xi d\eta$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \Bigg\{ \Bigg[-d_{16}s^2A_{mn}\ m\pi \sin(0)\sin(n\pi\eta) - d_{26}s^2RB_{mn}\ n\pi \sin(0)$$

$$\sin(n\pi\eta) + d_{66}(s^2RA_{mn}\ n\pi \cos(0)\cos(n\pi\eta) + s^2B_{mn}\ m\pi \cos(0)\cos(n\pi\eta))$$

$$- (4/3)f_{16}(-s^2A_{mn}\ m\pi \sin(0)\sin(n\pi\eta) - sC_{mn}\ m^2\pi^2 \sin(0)\sin(n\pi\eta))$$

$$- (4/3)f_{26}(-s^2RB_{mn}\ n\pi \sin(0)\sin(n\pi\eta) - sR^2C_{mn}\ n^2\pi^2 \sin(0)\sin(n\pi\eta))$$

$$- (4/3)f_{66}(s^2RA_{mn}\ n\pi \cos(0)\cos(n\pi\eta) + s^2B_{mn}\ m\pi \cos(0)\cos(n\pi\eta)$$

$$+2sRC_{mn}\ mn\pi^2 \cos(0)\cos(n\pi\eta)) + (4/3)f_{16}s^2A_{mn}\ m\pi \sin(0)\sin(n\pi\eta)$$

$$+ (4/3)f_{26}s^2RB_{mn}\ n\pi \sin(0)\sin(n\pi\eta) - (4/3)f_{66}(s^2RA_{mn}\ n\pi \cos(0)\cos(n\pi\eta)$$

$$+s^2B_{mn}\ m\pi \cos(0)\cos(n\pi\eta)) + (16/9)h_{16}(-s^2A_{mn}\ m\pi \sin(0)\sin(n\pi\eta)$$

$$-sC_{mn}\ m^2\pi^2 \sin(0)\sin(n\pi\eta)) + (16/9)h_{26}(-s^2RB_{mn}\ n\pi \sin(0)\sin(n\pi\eta)$$

$$-sR^2C_{mn}\ n^2\pi^2 \sin(0)\sin(n\pi\eta)) + (16/9)h_{66}(s^2RA_{mn}\ n\pi \cos(0)\cos(n\pi\eta)$$

$$+s^2B_{mn}\ m\pi\ \cos(0)\ \cos(n\pi\eta)+2sRC_{mn}\ mn\pi^2\ \cos(0)\ \cos(n\pi\eta))\Big]$$

$$\Big[\sin(0)\ \cos(q\pi\eta)\Big]$$

$$+\Big[d_{16}s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta)+d_{26}s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$-d_{66}(s^2RA_{mn}\ n\pi\ \cos(m\pi)\ \cos(n\pi\eta)+s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta))$$

$$-(4/3)f_{16}(s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta)+sC_{mn}\ m^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta))$$

$$-(4/3)f_{26}(s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta)+sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta))$$

$$-(4/3)f_{66}(-s^2RA_{mn}\ n\pi\ \cos(m\pi)\ \cos(n\pi\eta)-s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta)$$

$$-2sRC_{mn}\ mn\pi^2\ \cos(m\pi)\ \cos(n\pi\eta))-(4/3)f_{16}s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$-(4/3)f_{26}s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta)-(4/3)f_{66}(-s^2RA_{mn}\ n\pi\ \cos(m\pi)$$

$$\cos(n\pi\eta)-s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta))+(16/9)h_{16}(s^2A_{mn}\ m\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$+sC_{mn}\ m^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta))+(16/9)h_{26}(s^2RB_{mn}\ n\pi\ \sin(m\pi)\ \sin(n\pi\eta)$$

$$+sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi)\ \sin(n\pi\eta))+(16/9)h_{66}(-s^2RA_{mn}\ n\pi\ \cos(m\pi)\ \cos(n\pi\eta)$$

$$-s^2B_{mn}\ m\pi\ \cos(m\pi)\ \cos(n\pi\eta)-2sRC_{mn}\ mn\pi^2\ \cos(m\pi)\ \cos(n\pi\eta))\Big]$$

$$\Big[\sin(p\pi)\ \cos(q\pi n)\Big]\Big\}\,d\eta$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1\Big\{\Big[-d_{12}s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(0)-d_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)$$

$$\sin(0)+d_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\ \cos(0)+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0))$$

$$-(4/3)f_{12}(-s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(0)-sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(0))$$

$$-(4/3)f_{22}(-s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(0)-sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(0))$$

$$-(4/3)f_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\ \cos(0)+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0)$$

$$+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(0))+(8/3)f_{12}s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(0)$$

$$+(8/3)f_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(0)-(8/3)f_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\ \cos(0)$$

$$+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0))+(32/9)h_{12}(-s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(0)$$

$$-sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(0))+(32/9)h_{22}(-s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(0)$$

$$-sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(0)) + (32/9)h_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\ \cos(0)$$

$$+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0)+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(0))\ \Big]$$

$$\Big[\ \sin(p\pi\xi)\ \cos(0)\ \Big]$$

$$+\Big[\ d_{12}s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(n\pi) + d_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi)$$

$$+ d_{26}(-s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\ \cos(n\pi)-s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(n\pi))$$

$$- (4/3)f_{12}(s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(n\pi)+sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi))$$

$$- (4/3)f_{22}(s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi)+sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi))$$

$$- (4/3)f_{26}(-s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\ \cos(n\pi)-s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(n\pi)$$

$$-2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi)) - (8/3)f_{12}s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(n\pi)$$

$$- (8/3)f_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi) - (8/3)f_{26}(-s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)$$

$$\cos(n\pi)-s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(n\pi)) + (32/9)h_{12}(s^2A_{mn}\ m\pi\ \sin(m\pi\xi)\ \sin(n\pi)$$

$$+sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi)) + (32/9)h_{22}(s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi)$$

$$+sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi)) + (32/9)h_{26}(-s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\ \cos(n\pi)$$

$$-s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(n\pi)+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi))\ \Big]$$

$$\Big[\ \sin(p\pi\xi)\ \cos(q\pi)\ \Big]\ \Big\}\ d\xi = 0 \qquad (109)$$

We now will simplify the above by using $\sin(0) = \sin(m\pi) = 0$ and $\cos(0) = 1$ to yield

$$\sum_{m=1}^{\infty}\ \sum_{n=1}^{\infty}\ \int_0^1\!\!\int_0^1 \Big\{ -d_{12}s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)$$

$$- d_{22}s^2R^2B_{mn}n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta) + d_{26}(-s^2R^2A_{mn}\ n^2\pi^2$$

$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta)-s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)$$

$$\cos(q\pi\eta)) - (4/3)f_{12}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)$$

$$-sRC_{mn}\ m^2 n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{22}(-s^2R^2B_{mn}$$
$$n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta) - sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\cos(q\pi\eta) - s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta)$$
$$-2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta)) - d_{16}s^2A_{mn}\ m^2\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) - d_{26}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\cos(q\pi\eta) + d_{66}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)$$
$$-sB_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{16}(-s^2A_{mn}\ m^2\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) - sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{26}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)$$
$$\cos(q\pi\eta) - s^2R^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) - (4/3)f_{66}$$
$$(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta) - s^2B_{mn}\ m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta) - 2s^2RC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta)) - (\ a_{44} - 8d_{44} + 16f_{44}\ )(s^4B_{mn}\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta) + s^3RC_{mn}\ n\pi\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta))$$
$$-(\ a_{45} - 8d_{45} + 16f_{45}\ )(s^4A_{mn}\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) + s^3C_{mn}\ m\pi$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta)) + (4/3)f_{12}s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta) + (4/3)f_{22}s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta) - (4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)$$
$$\cos(q\pi\eta) - s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta))$$
$$+ (16/9)h_{12}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta) - sRC_{mn}\ m^2n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{22}(-s^2R^2B_{mn}\ n^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta) - sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\cos(q\pi\eta) - s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta)$$
$$-2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta)) + (4/3)f_{16}s^2A_{mn}\ m^2\pi^2$$

$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) + (4/3)f_{26}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)$$

$$\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) + (4/3)f_{66}(s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\cos(q\pi\eta) - s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)) +$$

$$(16/9)h_{16}(-s^2A_{mn}\ m^2\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) - sC_{mn}\ m^3\pi^3$$

$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{26}(-s^2RB_{mn}\ mn\pi^2$$

$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\cos(q\pi\eta) - s^2R^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$\sin(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{66}(-s^2RA_{mn}\ mn\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)$$

$$\cos(q\pi\eta) - s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta) - 2sRC_{mn}\ m^2n\pi^3$$

$$\sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\cos(q\pi\eta)) + \omega^2\ (\ 17/315s^2B_{mn}\ \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\cos(q\pi\eta)\ -\ 4/315sC_{mn}\ n\pi\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)) \Big\}\ d\xi d\eta$$

$$+ \sum_{m=1}^{\infty}\ \sum_{n=1}^{\infty} \int_0^1 \Big\{\ d_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\sin(p\pi\xi) + s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi))$$

$$-\ (4/3)f_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\sin(p\pi\xi) + s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)) - (8/3)f_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)) + (32/9)h_{26}(s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi) + 2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi))$$

$$+\ d_{26}(-s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi) - s^2B_{mn}\ m\pi\ \cos(m\pi\xi)$$

$$\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)) - (4/3)f_{26}(-s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi)\cos(q\pi) - s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)$$

$$-2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)) - (8/3)f_{26}(-s^2RA_{mn}\ n\pi$$

$$\cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi) - s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi)\cos(q\pi)) + (32/9)h_{26}(-s^2RA_{mn}\ n\pi\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi)\cos(q\pi) - s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)$$

$$+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)) \Big\}\ d\xi = 0 \qquad (110)$$

Using integration notation we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg\{\Bigg[(1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2Rmn\pi^2)(-d_{12}+8/3f_{12}-d_{66}+4/3f_{26}+4/3f_{66}$$

$$-16/9h_{12}-16/9h_{66}) + (0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})$$

$$(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^2R^2n^2\pi^2)(-d_{26}+8/3f_{26}+4/3f_{66}-16/9h_{26})$$

$$(0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^2m^2\pi^2)$$

$$(-d_{16}+8/3f_{16}-16/9h_{16})(0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})$$

$$(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^4)(-a_{45}+8d_{45}-16f_{45})$$

$$(0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(1-\cos(n\pi)$$

$$\cos(q\pi))(s^2Rn\pi)(d_{26}-4f_{26}+32/9h_{26})\Bigg]A_{mn}$$

$$+\Bigg[(1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2R^2n^2\pi^2)(-d_{22}+8/3f_{22}-16/9h_{22}) + (1/2 \text{ or } 0)$$

$$(1/2 \text{ or } 0)(s^2m^2\pi^2)(-d_{66}+4/3f_{66}-16/9h_{66}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^4)$$

$$(-a_{44}+8d_{44}-16f_{44}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(17/315\bar{\omega}^2s^2) + (0 \text{ or } 0 \text{ even},$$

$$2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^2Rmn\pi^2)(-2d_{26}+4f_{26}$$

$$-32/9h_{26}+4/3f_{66}) + (0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},$$

$$2n/\pi(n^2-q^2) \text{ odd})(1-\cos(n\pi)\cos(q\pi))(s^2m\pi)((d_{26}-4f_{26}+32/9h_{26})\Bigg]B_{mn}$$

$$+\Bigg[(1/2 \text{ or } 0)(1/2 \text{ or } 0)(sRm^2n\pi^3)(4/3f_{12}+8/3f_{66}-16/9h_{12}-32/9h_{66})$$

$$+ (1/2 \text{ or } 0)(1/2 \text{ or } 0)(sR^3n^3\pi^3)(4/3f_{22}-16/9h_{22}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)$$

$$(s^3Rn\pi)(-a_{44}+8d_{44}-16f_{44}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(-4/315\bar{\omega}^2sn\pi)$$

$$(0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(sR^2mn^2\pi^3)$$

$$(4f_{26}-48/9h_{26}) + (0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},$$

$$2n/\pi(n^2-q^2) \text{ odd})(sm^3\pi^3)(4/3f_{16}-16/9h_{16})(0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})$$

$$(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^3m\pi)(-a_{45}+8d_{45}-16f_{45})(0 \text{ or } 0 \text{ even},$$

$$2p/\pi(p^2-m^2) \text{ odd})(1-\cos(n\pi)\cos(q\pi))(sRmn\pi^2)(-8/3f_{26}+64/9h_{26})\Bigg]C_{mn}\Bigg\} = 0 \tag{111}$$

Clamped Boundary Condition A plate which is clamped on all four sides has the following requirements

@ $x = 0, a$:

$$w = \psi_x = \psi_y = 0 \tag{112}$$

and

@ $y = 0, b$:

$$w = \psi_x = \psi_y = 0 \tag{113}$$

We choose the following set of admissible functions

$$\begin{aligned}
\psi_x &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \\
\psi_y &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \\
w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi x/a) \sin(n\pi y/b)
\end{aligned} \tag{114}$$

Normalizing Eq.(114) we obtain

$$\begin{aligned}
\psi_\xi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi\xi) \sin(n\pi\eta) \\
\psi_\eta &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi\xi) \sin(n\pi\eta) \\
w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi\xi) \sin(n\pi\eta)
\end{aligned} \tag{115}$$

We next calculate the needed derivatives

$$\psi_{\xi,\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, m\pi \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\ m^3\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\ m^2n\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\ mn^2\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\ n\pi \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\ n^3\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\ n\pi \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ n^3\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\ mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ m^2n\pi^3 \sin(m\pi\xi) \cos(n\pi\eta) \qquad (116)$$

$$\psi_{\eta',\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ mn^2\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\ m\pi \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\ m^3\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ m\pi \cos(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ m^4\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ n\pi \sin(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ n^3\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\eta\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ n^4\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ m^3 n\pi^4 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$w_{,\xi\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\ m^2 n^2\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$w_{,\xi\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\ mn^3\pi^4 \cos(m\pi\xi)\cos(n\pi\eta)$$

For the δw equation of motion we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\!\!\int_0^1 \Big\{ (a_{45}-8d_{45}+16f_{45})(s^3 B_{mn}\ m\pi \cos(m\pi\xi)\sin(n\pi\eta)$$
$$+s^3 RA_{mn}\ n\pi \sin(m\pi\xi)\cos(n\pi\eta)+2s^2 RC_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta))$$
$$+ (a_{55}-8d_{55}+16f_{55})(s^3 A_{mn}\ m\pi \cos(m\pi\xi)\sin(n\pi\eta)-s^2 C_{mn}\ m^2\pi^2$$
$$\sin(m\pi\xi)\sin(n\pi\eta)) + (a_{44}-8d_{44}+16f_{44})(s^3 RB_{mn}\ n\pi \sin(m\pi\xi)\cos(n\pi\eta)$$
$$-s^2 R^2 C_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)) + N_0 k_1 C_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$
$$+ N_0 R^2 k_2 C_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) + N_0 Rk_3 C_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta))$$
$$- (4/3)f_{11} sA_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) - (4/3)sRf_{12}$$

$$
\begin{aligned}
&B_{mn}\, m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) + (4/3)f_{16}(-sRA_{mn}\, m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) \\
&-sB_{mn}\, m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) - (16/9)h_{11}(-sA_{mn}\, m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) \\
&+C_{mn}\, m^4\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)) - (16/9)h_{12}(-sRB_{mn}\, m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) \\
&+R^2C_{mn}\, m^2n^2\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)) - (16/9)h_{16}(-sRA_{mn}\, m^2 n\pi^3 \\
&\sin(m\pi\xi)\cos(n\pi\eta) - sB_{mn}\, m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) - 2RC_{mn}\, m^3 n\pi^4 \\
&\cos(m\pi\xi)\cos(n\pi\eta)) - (4/3)sR^2 f_{12}A_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) \\
&- (4/3)sR^3 f_{22}B_{mn}\, n^3\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) + (4/3)f_{26}(-sR^3A_{mn}\, n^3\pi^3 \\
&\sin(m\pi\xi)\cos(n\pi\eta) - sR^2B_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) \\
&- (16/9)h_{12}(-sR^2A_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) + R^2C_{mn}\, m^2n^2\pi^4 \\
&\sin(m\pi\xi)\sin(n\pi\eta)) - (16/9)h_{22}(-sR^3B_{mn}\, n^3\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) \\
&+R^4C_{mn}\, n^4\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)) - (16/9)h_{26}(-sR^3A_{mn}\, n^3\pi^3 \\
&\sin(m\pi\xi)\cos(n\pi\eta) - sR^2B_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) - 2R^3C_{mn}\, mn^3\pi^4 \\
&\cos(m\pi\xi)\cos(n\pi\eta)) - (8/3)sRf_{16}A_{mn}\, m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) \\
&- (8/3)sR^2 f_{26}B_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) + (8/3)f_{66}(-sR^2A_{mn}\, mn^2\pi^3 \\
&\cos(m\pi\xi)\sin(n\pi\eta) - sRB_{mn}\, m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)) \\
&- (32/9)h_{16}(-sRA_{mn}\, m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) - RC_{mn}\, m^3 n\pi^4 \cos(m\pi\xi)\cos(n\pi\eta)) \\
&- (32/9)h_{26}(-sR^2B_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) - R^3C_{mn}\, mn^3\pi^4 \\
&\cos(m\pi\xi)\cos(n\pi\eta)) - (32/9)h_{66}(-sR^2A_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta) \\
&-sRB_{mn}\, m^2 n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta) + 2R^2C_{mn}\, m^2n^2\pi^4 \sin(m\pi\xi)\sin(n\pi\eta)) \\
&+ \omega^2\,(\, s^2C_{mn} \sin(m\pi\xi)\sin(n\pi\eta) + 16/4032C_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) \\
&+16/4032R^2C_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) + 4/315sA_{mn}\, m\pi \cos(m\pi\xi)\sin(n\pi\eta) \\
&+4/315sRB_{mn}\, n\pi \sin(m\pi\xi)\cos(n\pi\eta)\,)\Big\}\Big\{\sin(p\pi\xi)\sin(q\pi\eta)\Big\}\, d\xi d\eta = 0
\end{aligned}
\tag{117}
$$

where by inspection, the boundary terms are zero.

We now multiply through by $\sin(p\pi\xi)\sin(q\pi\eta)$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\int_0^1 \Big\{ (a_{45}-8d_{45}+16f_{45})(s^3B_{mn}\ m\pi \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)$$

$$\sin(q\pi\eta)+s^3RA_{mn}\ n\pi \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)+2s^2RC_{mn}\ mn\pi^2$$

$$\cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) + (a_{55}-8d_{55}+16f_{55})$$

$$(s^3A_{mn}\ m\pi \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-s^2C_{mn}\ m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)$$

$$\sin(n\pi\eta)\sin(q\pi\eta)) + (a_{44}-8d_{44}+16f_{44})(s^3RB_{mn}\ n\pi \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\sin(q\pi\eta)-s^2R^2C_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta))$$

$$+ N_0k_1C_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)+ N_0R^2k_2C_{mn}\ n^2\pi^2$$

$$\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) + N_0Rk_3C_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{11}sA_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)$$

$$\sin(n\pi\eta)\sin(q\pi\eta) - (4/3)sRf_{12}B_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)$$

$$+ (4/3)f_{16}(-sRA_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)$$

$$-sB_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{11}(-sA_{mn}\ m^3\pi^3$$

$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)+C_{mn}\ m^4\pi^4 \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{12}(-sRB_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\sin(q\pi\eta)+R^2C_{mn}\ m^2n^2\pi^4 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta))$$

$$- (16/9)h_{16}(-sRA_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)$$

$$-sB_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - 2RC_{mn}\ m^3n\pi^4$$

$$\cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)sR^2f_{12}A_{mn}\ mn^2\pi^3$$

$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - (4/3)sR^3f_{22}B_{mn}\ n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{26}(-sR^3A_{mn}\ n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\sin(q\pi\eta)-sR^2B_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta))$$

$$- (16/9)h_{12}(-sR^2A_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)+R^2C_{mn}\ m^2n^2\pi^4$$

$$\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{22}(-sR^3B_{mn}\ n^3\pi^3$$

$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)+R^4C_{mn}\ n^4\pi^4 \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{26}(-sR^3A_{mn}\ n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi\eta)\sin(q\pi\eta)-sR^2B_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$

$-2R^3C_{mn}\ mn^3\pi^4\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) - (8/3)sRf_{16}A_{mn}\ m^2n\pi^3$

$\sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - (8/3)sR^2f_{26}B_{mn}\ mn^2\pi^3$

$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) + (8/3)f_{66}(-sR^2A_{mn}\ mn^2\pi^3$

$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)-sRB_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$

$\cos(n\pi\eta)\sin(q\pi\eta)) - (32/9)h_{16}(-sRA_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$

$\cos(n\pi\eta)\sin(q\pi\eta)-RC_{mn}\ m^3n\pi^4\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta))$

$- (32/9)h_{26}(-sR^2B_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)-R^3C_{mn}\ mn^3\pi^4$

$\cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) - (32/9)h_{66}(-sR^2A_{mn}\ mn^2\pi^3$

$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)-sRB_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$

$\cos(n\pi\eta)\sin(q\pi\eta)+2R^2C_{mn}\ m^2n^2\pi^4\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)$

$+ \omega^2\ (\ s^2C_{mn}\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+16/4032C_{mn}\ m^2\pi^2$

$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+16/4032R^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)$

$\sin(n\pi\eta)\sin(q\pi\eta)+4/315sA_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)$

$$+4/315sRB_{mn}\ n\pi\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)\ )\ \Big\}\ d\xi d\eta = 0 \qquad (118)$$

We now simplify the integration of the previous equation. The same integration scheme as shown in the simply supported boundary condition will be used here. Thus

$\sum_{m=1}^{\infty}\ \sum_{n=1}^{\infty} \Big\{ \Big[$ (0 or 0 even,$2p/\pi(p^2-m^2)$ odd)(1/2or0)$(s^3m\pi)(a_{55}-8d_{55}+16f_{55})$

+ (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(1/2 or 0)$(sm^3\pi^3)(-4/3f_{11}+16/9h_{11})$

+ (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(1/2 or 0)$(sR^2mn^2\pi^3)(-4/3f_{12}+16/9h_{12}$

$-8/3f_{66}+32/9h_{66})$ + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(1/2 or 0)

$(4/315\omega^{-2}sm\pi)$ + (1/2 or 0)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)$(s^3Rn\pi)$

$(a_{45}-8d_{45}+16f_{45})$ + (1/2 or 0)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)$(sRm^2n\pi^3)$

$(-4f_{16}+48/9h_{16})$ + (1/2 or 0)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)

$(sR^3n^3\pi^3)(-4/3f_{26}+16/9h_{26})\Big]\ A_{mn}$

+ $\Big[$ (1/2 or 0)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)$(s^3Rn\pi)(a_{44}-8d_{44}+16f_{44})$

+ (1/2 or 0)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)$(sRm^2n\pi^3)$

$(-4/3f_{12}+16/9h_{12}-8/3f_{66}+32/9h_{26})$ + (1/2 or 0)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)$(sR^3n^3\pi^3)(-4/3f_{22}+16/9h_{22})$ + (1/2 or 0)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)$(-4/315\bar{\omega}^2sRn\pi)$ + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)

(1/2 or 0)$(s^3m\pi)(a_{45}-8d_{45}+16f_{45})$ + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)

(1/2 or 0)$(sm^3\pi^3)(16/9h_{16})$ + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)

(1/2 or 0)$(sR^2mn^2\pi^3)(-4f_{26}+48/9h_{26})\Big]\ B_{mn}$

+ $\Big[$ (1/2 or 0)(1/2 or 0)$(s^2m^2\pi^2)(-a_{55}+8d_{55}-16f_{55})$ + (1/2 or 0)

(1/2 or 0)$(s^2R^2n^2\pi^2)(-a_{44}+8d_{44}-16f_{44})$ + (1/2 or 0)(1/2 or 0)$(\bar{N}_0k_1m^2\pi^2$

$+\bar{N}_0R^2k_2n^2\pi^2+\bar{\omega}^2s^2+16/4032\bar{\omega}^2m^2\pi^2+16/4032\bar{\omega}^2R^2n^2\pi^2)$ + (1/2 or 0)(1/2 or 0)

$(\pi^4)(-16/9h_{11}m^4-32/9h_{12}R^2m^2n^2-16/9h_{22}R^4n^4-64/9h_{66}R^2m^2n^2)$ +

(0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)$(2s^2Rmn\pi^2)$

$(a_{45}-8d_{45}+16f_{45})$ + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)$(\bar{N}_0Rk_3mn\pi^2)$ + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)

(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)$(64/9\pi^4)(h_{16}Rm^3n+h_{26}R^3mn^3)\Big]\ C_{mn}\Big\} = 0$

(119)

These equations will be programmed using the same technique as seen in the simply supported boundary condition.

For the $\delta\psi_x$ equation of motion we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 \Big\{ -d_{11} s^2 A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) + d_{12} s^2 RB_{mn}\ mn\pi^2$$

$\cos(m\pi\xi)\ \cos(n\pi\eta) + d_{16}(s^2 RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-s^2 B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{11}(-s^2 A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$-sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{12}(-s^2 RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-sR^2 C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{16}(s^2 RA_{mn}\ mn\pi^2$

$\cos(m\pi\xi)\ \cos(n\pi\eta) - s^2 B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) - 2sRC_{mn}\ m^2 n\pi^3$

$\sin(m\pi\xi)\ \cos(n\pi\eta)) + d_{16} s^2 RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$- d_{26} s^2 R^2 B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) + d_{66}(-s^2 R^2 A_{mn}\ n^2\pi^2$

$\cos(m\pi\xi)\ \cos(n\pi\eta) + s^2 RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)) - (4/3)f_{16}(s^2 RA_{mn}$

$mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) - sRC_{mn}\ m^2 n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$- (4/3)f_{26}(-s^2 R^2 B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) - sR^3 C_{mn}\ n^3\pi^3$

$\sin(m\pi\xi)\ \cos(n\pi\eta)) - (4/3)f_{66}(-s^2 R^2 A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$+s^2 RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) - 2sR^2 C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta))$

$- (\ a_{45} - 8d_{45} + 16f_{45}\ )(s^4 B_{mn}\ \sin(m\pi\xi)\ \sin(n\pi\eta) + s^3 RC_{mn}\ n\pi$

$\sin(m\pi\xi)\ \cos(n\pi\eta)) - (\ a_{55} - 8d_{55} + 16f_{55}\ )(s^4 A_{mn}\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$+s^3 C_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi\eta)) + (4/3)f_{11} s^2 A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$- (4/3)\ f_{12} s^2 RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) - (4/3)f_{16}(s^2 RA_{mn}\ mn\pi^2$

$\cos(m\pi\xi)\ \cos(n\pi\eta) - s^2 B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)) + (16/9)h_{11}$

$(-s^2 A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) - sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta))$

$+ (16/9)h_{12}(s^2 RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) - sR^2 C_{mn}\ mn^2\pi^3$

$\cos(m\pi\xi)\ \sin(n\pi\eta)) + (16/9)h_{16}(s^2 RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-s^2 B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) - 2sRC_{mn}\ m^2 n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$- (4/3)f_{16} s^2 RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) + (4/3)f_{26} s^2 R^2 B_{mn}\ n^2\pi^2$

$\sin(m\pi\xi)\ \sin(n\pi\eta) + (4/3)f_{66}(s^2 R^2 A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$-s^2 RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)) + (16/9)h_{16}(s^2 RA_{mn}\ mn\pi^2$

$\cos(m\pi\xi)\ \cos(n\pi\eta) - sRC_{mn}\ m^2 n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta)) + (16/9)h_{26}(-s^2 R^2 B_{mn}$

$$n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) - sR^3C_{mn}\, n^3\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)) + (16/9)h_{66}$$
$$(-s^2R^2A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) + s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$
$$-2sR^2C_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) + 17/315\omega^2 s^2 A_{mn} \sin(m\pi\xi)\sin(n\pi\eta)$$
$$- 4/315\bar{\omega}^2 sC_{mn}\, m\pi \cos(m\pi\xi)\sin(n\pi\eta) \Big\} \Big\{ \sin(p\pi\xi)\sin(q\pi\eta) \Big\}\, d\xi d\eta = 0 \tag{120}$$

where the boundary terms are seen to be zero.

We now multiply through by $\{\sin(p\pi\xi)\sin(q\pi\eta)\}$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1\int_0^1 \Big\{ -d_{11}s^2A_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) + d_{12}$$
$$s^2RB_{mn}mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) + d_{16}(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - s^2B_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta))$$
$$- (4/3)f_{11}(-s^2A_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) - sC_{mn}\, m^3\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{12}(s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - sR^2C_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta) - s^2B_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta) - 2sRC_{mn}$$
$$m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) + d_{16}s^2RA_{mn}\, mn\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - d_{26}s^2R^2B_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta) + d_{66}(-s^2R^2A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$
$$+s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}(s^2RA_{mn}$$
$$mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - sRC_{mn}\, m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{26}(-s^2R^2B_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta) - sR^3C_{mn}\, n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (4/3)f_{66}(-s^2R^2A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$
$$+s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta) - 2sR^2C_{mn}\, mn^2\pi^3$$

$$\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)) - (a_{45}-8d_{45}+16f_{45})(s^4B_{mn} \sin(m\pi\xi)$$
$$\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)+s^3RC_{mn} n\pi \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (a_{55}-8d_{55}+16f_{55})(s^4A_{mn} \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$
$$+s^3C_{mn} m\pi \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (4/3)f_{11}s^2A_{mn} m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - (4/3) f_{12}s^2RB_{mn} mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) - (4/3)f_{16}(s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)-s^2B_{mn} m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta))$$
$$+ (16/9)h_{11}(-s^2A_{mn} m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$
$$-sC_{mn} m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{12}(s^2RB_{mn} mn\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-sR^2C_{mn} mn^2\pi^3\cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{16}(s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)$$
$$\sin(q\pi\eta)-s^2B_{mn} m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-2sRC_{mn} m^2n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{26}s^2R^2B_{mn} n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{66}(s^2R^2A_{mn} n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)$$
$$\sin(q\pi\eta)-s^2RB_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{16}$$
$$(s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-sRC_{mn} m^2n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{26}(-s^2R^2B_{mn}n^2\pi^2 \sin(m\pi\xi)$$
$$\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-sR^3C_{mn} n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$
$$+ (16/9)h_{66}(-s^2R^2A_{mn} n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)+s^2RB_{mn} mn\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-2sR^2C_{mn} mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) + 17/315\omega^2s^2A_{mn} \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$
$$- 4/315\bar{\omega}^2sC_{mn} m\pi \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) \Big\} d\xi\, d\eta = 0 \qquad (121)$$

Using the integration notation

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg\{ \Bigg[(1/2 \text{ or } 0)(1/2 \text{ or} 0)(s^2 m^2 \pi^2)(-d_{11}+8/3f_{11}-16/9h_{11}) + (1/2 \text{ or} 0)$$

$$(1/2 \text{ or } 0)(s^2 R^2 n^2 \pi^2)(4/3f_{16}-d_{66}+8/3f_{66}-16/9h_{66}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)$$

$$(s^4)(-a_{55}+8d_{55}-16f_{55}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(17/315\bar{\omega}^2 s^2)$$

$$(0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even, } 2q/\pi(q^2-n^2) \text{ odd})(s^2 Rmn\pi^2)$$

$$(2d_{16}-4/3f_{12}-16/3f_{16}+32/9h_{16}+16/9h_{66}) \Bigg] A_{mn}$$

$$+ \Bigg[(0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even, } 2q/\pi(q^2-n^2) \text{ odd})$$

$$(s^2 Rmn\pi^2)(d_{12}+d_{66}-8/3f_{66}-4/3f_{12}+16/9h_{12}-16/9h_{66}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)$$

$$(s^2 m^2 \pi^2)(-d_{16}+4/3f_{16}-16/9h_{16}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^4)$$

$$(-a_{45}+8d_{45}-16f_{45}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2 R^2 n^2 \pi^2)$$

$$(-d_{26}+8/3f_{26}-16/9h_{26}) \Bigg] B_{mn}$$

$$+ \Bigg[(0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})(1/2 \text{ or } 0)(sm^3\pi^3)(-16/9h_{11})$$

$$(0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})(1/2 \text{ or } 0)(sR^2 mn^2 \pi^3)$$

$$(4/3f_{12}+8/3f_{66}-16/9h_{12}-32/9h_{66}) + (0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})$$

$$(1/2 \text{ or } 0)(sm^3\pi)(-a_{55}+8d_{55}-16f_{55}) + (0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})$$

$$(1/2 \text{ or } 0)(-4/315\bar{\omega}^2 sm\pi) + (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even, } 2q/\pi(q^2-n^2) \text{ odd})$$

$$(sRm^2 n\pi^3)(4f_{16}-48/9h_{16}) + (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even, } 2q/\pi(q^2-n^2) \text{ odd})$$

$$(sR^3 n^3 \pi^3)(4/3f_{26}-16/9h_{26}) + (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even, } 2q/\pi(q^2-n^2) \text{ odd})$$

$$(s^3 Rn\pi)(-a_{45}+8d_{45}-16f_{45}) \Bigg] C_{mn} \Bigg\} = 0 \tag{122}$$

For the $\delta\psi_y$ equation of motion we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\!\!\int_0^1 \Bigg\{ d_{12} s^2 RA_{mn}\, mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta) - d_{22} s^2 R^2 B_{mn}\, n^2\pi^2$$

$$\sin(m\pi\xi) \sin(n\pi\eta) + d_{26}(-s^2 R^2 A_{mn}\, n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)+s^2 RB_{mn}\, mn\pi^2$$

$\cos(m\pi\xi)\ \cos(n\pi\eta)) - (4/3)f_{12}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta)) - (4/3)f_{22}(-s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)$

$\sin(n\pi\eta)-sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$- (4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)+s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - d_{16}s^2A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$+ d_{26}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) + d_{66}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-sB_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{16}(-s^2A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$-sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{26}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-s^2R^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta) - (4/3)f_{66}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)$

$\cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)-2s^2RC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$- (\ a_{44}-8d_{44}+16f_{44}\ )(s^4B_{mn}\ \sin(m\pi\xi)\ \sin(n\pi\eta)+s^3RC_{mn}\ n\pi\ \sin(m\pi\xi)$

$\cos(n\pi\eta)) - (\ a_{45}-8d_{45}+16f_{45}\ )(s^4A_{mn}\ \sin(m\pi\xi)\ \sin(n\pi\eta)+s^3C_{mn}\ m\pi$

$\cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{12}s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$+ (4/3)f_{22}s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) - (4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2$

$\sin(m\pi\xi)\ \sin(n\pi\eta)+s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta))$

$+ (16/9)h_{12}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)-sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$+ (16/9)h_{22}(-s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)-sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$+ (16/9)h_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)+s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{16}s^2A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$- (4/3)f_{26}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta) + (4/3)f_{66}(-s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)$

$\cos(n\pi\eta)+s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)) + (16/9)h_{16}(-s^2A_{mn}\ m^2\pi^2$

$\sin(m\pi\xi)\ \sin(n\pi\eta)-sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) + (16/9)h_{26}(s^2RB_{mn}\ mn\pi^2$

$\cos(m\pi\xi)\ \cos(n\pi\eta)-s^2R^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta))$

$+ (16/9)h_{66}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$-2sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta)) + \omega^2\ (\ 17/315s^2B_{mn}\ \sin(m\pi\xi)$

$\sin(n\pi\eta) - 4/315sC_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi\eta))\ \}$

$$\left\{ \sin(p\pi\xi)\ \sin(q\pi\eta) \right\} d\xi d\eta = 0 \qquad (123)$$

where the boundaries go to zero.

We now will simplify the above by multiplying through with $\{ \sin(p\pi\xi)\ \sin(q\pi\eta) \}$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\int_0^1 \Big\{ d_{12}s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta) - d_{22}$$
$$s^2R^2B_{mn}n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) + d_{26}(-s^2R^2A_{mn}\ n^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)$$
$$\sin(q\pi\eta)) - (4/3)f_{12}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)$$
$$-sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{22}(-s^2R^2B_{mn}$$
$$n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)-sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)+s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)$$
$$-2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)) - d_{16}s^2A_{mn}\ m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) + d_{26}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta) + d_{66}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)$$
$$-sB_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}(-s^2A_{mn}\ m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)-sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{26}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)$$
$$\sin(q\pi\eta)-s^2R^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta) - (4/3)f_{66}$$
$$(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)-s^2B_{mn}\ m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)-2s^2RC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)) - (\ a_{44}-8d_{44}+16f_{44}\ )(s^4B_{mn}\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)+s^3RC_{mn}\ n\pi\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta))$$

$$- \left(a_{45}-8d_{45}+16f_{45} \right)(s^4A_{mn} \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)+s^3C_{mn} m\pi \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{12}s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{22}s^2R^2B_{mn} n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - (4/3)f_{26}(-s^2R^2A_{mn} n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta) \sin(q\pi\eta)+s^2RB_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$

$$+ (16/9)h_{12}(s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-sRC_{mn} m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{22}(-s^2R^2B_{mn} n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-sR^3C_{mn} n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{26}(-s^2R^2A_{mn} n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)+s^2RB_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) -2sR^2C_{mn} mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (4/3)f_{16}s^2A_{mn} m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - (4/3)f_{26}s^2RB_{mn} mn\pi^2 \cos(m\pi\xi) \sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{66}(-s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)+s^2B_{mn} m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{16}(-s^2A_{mn} m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-sC_{mn} m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{26}(s^2RB_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-s^2R^2C_{mn} mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{66}(s^2RA_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta) \sin(q\pi\eta)-s^2B_{mn} m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-2sRC_{mn} m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))+ \omega^2 \left(17/315s^2B_{mn} \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - 4/315sC_{mn} n\pi \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) \right\} d\xi d\eta = 0 \qquad (124)$$

Using the integration notation we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Bigg\{ \Bigg[(0 \text{ or } 0 \text{even}, \ 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even}, \ 2q/\pi(q^2-n^2) \text{ odd})$$

$$(s^2Rmn\pi^2)(d_{12}-8/3f_{12}+d_{66}-4/3f_{26}-8/3f_{66}+16/9h_{12}+16/9h_{66}) + (1/2 \text{ or } 0)$$

$$(1/2 \text{ or } 0)(s^2R^2n^2\pi^2)(-d_{26}+8/3f_{26}+4/3f_{66}-16/9h_{26}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)$$

$$(s^4)(-a_{45}+8d_{45}-16f_{45}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2m^2\pi^2)$$

$$(-d_{16}+8/3f_{16}-16/9h_{16}) \Bigg] A_{mn}$$

$$+ \Bigg[(0 \text{ or } 0 \text{ even}, \ 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even}, \ 2q/\pi(q^2-n^2) \text{ odd})$$

$$(s^2Rmn\pi^2)(2d_{26}-4f_{26}+32/9h_{26}+4/3f_{66}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2R^2n^2\pi^2)$$

$$(-d_{22}+8/3f_{22}-16/9h_{22}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2m^2\pi^2)$$

$$(-d_{66}+4/3f_{66}-16/9h_{66}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^4)(-a_{44}+8d_{44}-16f_{44})$$

$$+ (1/2 \text{ or } 0)(1/2 \text{ or } 0)(17/315\bar{\omega}^2s^2) \Bigg] B_{mn}$$

$$+ \Bigg[(1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even}, \ 2q/\pi(q^2-n^2) \text{ odd})(sRm^2n\pi^3)$$

$$(4/3f_{12}-16/9h_{12}-32/9h_{66}) + (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even}, \ 2q/\pi(q^2-n^2) \text{ odd})$$

$$(sR^3n^3\pi^3)(4/3f_{22}-16/9h_{22}) + (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even}, \ 2q/\pi(q^2-n^2) \text{ odd})$$

$$(s^3Rn\pi)(-a_{44}+8d_{44}-16f_{44}) + (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even}, \ 2q/\pi(q^2-n^2) \text{ odd})$$

$$(-4/315\bar{\omega}^2sn\pi) + (0 \text{ or } 0 \text{ even}, \ 2q/\pi(q^2-n^2) \text{ odd})(1/2 \text{ or } 0)(sR^2mn^2\pi^3)$$

$$(4f_{26}+8/3f_{66}-48/9h_{26}) + (0 \text{ or } 0 \text{ even}, \ 2p/\pi(p^2-m^2) \text{ odd})(1/2 \text{ or } 0)$$

$$(sm^3\pi^3)(4/3f_{16}-16/9h_{16})(0 \text{ or } 0 \text{ even}, \ 2p/\pi(p^2-m^2) \text{ odd})(1/2 \text{ or } 0)$$

$$(s^3m\pi)(-a_{45}+8d_{45}-16f_{45}) \Bigg] \Bigg\} C_{mn} = 0 \tag{125}$$

These equations will be programmed as before.

Clamped - Simply Supported Boundary Condition For a plate clamped on two opposite sides and simply supported on the two other sides, the following conditions exist

@ $x = 0, a$:

$$w = \psi_x = \psi_y = 0 \tag{126}$$

@ $y = 0, b$:

$$w = \psi_x = 0$$

$$M_y = D_{12}\psi_{x},_{x} + D_{22}\psi_{y},_{y} + D_{26}(\psi_{x},_{y}+\psi_{y},_{x}) - (4/3h^2)F_{12}(\psi_{x},_{x}+w,_{xx}) - (4/3h^2)F_{22}(\psi_{y},_{y}+w,_{yy}) - (4/3h^2)F_{26}(\psi_{x},_{y}+\psi_{y},_{x}+2w,_{xy}) = 0 \tag{127}$$

where the clamped boundaries are along $x = 0$ and $x = a$, and the simply supported boundaries are along $y = 0$ and $y = b$. We choose the following admissible functions

$$\begin{aligned}
\psi_x &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \\
\psi_y &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \\
w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi x/a) \sin(n\pi y/b)
\end{aligned} \tag{128}$$

Normalizing Eq.(128)

$$\begin{aligned}
\psi_\xi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi\xi) \sin(n\pi\eta) \\
\psi_\eta &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin(m\pi\xi) \cos(n\pi\eta) \\
w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi\xi) \sin(n\pi\eta)
\end{aligned} \tag{129}$$

We next calculate the needed derivatives

$$\psi_{\xi',\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, m\pi \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, m^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, m^3\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, m^2n\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, mn^2\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, n\pi \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\xi',\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\xi',\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -A_{mn}\, n^3\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, n\pi \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, n^2\pi^2 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\, n^3\pi^3 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, mn\pi^2 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$\psi_{\eta',\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\, m^2n\pi^3 \sin(m\pi\xi) \sin(n\pi\eta) \tag{130}$$

$$\psi_{\eta',\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, mn^2\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\, m\pi \cos(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, m^2\pi^2 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$\psi_{\eta',\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -B_{mn}\, m^3\pi^3 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$w_{,\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\, m\pi \cos(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, m^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, m^3\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\xi\xi\xi} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\, m^4\pi^4 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\, mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$w_{,\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, m^2n\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$w_{,\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, mn^2\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\, n\pi \sin(m\pi\xi) \cos(n\pi\eta)$$

$$w_{,\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, n^3\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)$$

$$w_{,\eta\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\, n^4\pi^4 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\xi\xi\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, m^3n\pi^4 \cos(m\pi\xi) \cos(n\pi\eta)$$

$$w_{,\xi\xi\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}\, m^2n^2\pi^4 \sin(m\pi\xi) \sin(n\pi\eta)$$

$$w_{,\xi\eta\eta\eta} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -C_{mn}\, mn^3\pi^4 \cos(m\pi\xi) \cos(n\pi\eta)$$

For the δw equation of motion we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\!\!\int_0^1 \Big\{ (a_{45}-8d_{45}+16f_{45})(s^3B_{mn}\, m\pi \cos(m\pi\xi) \cos(n\pi\eta)$$

$$+s^3RA_{mn}\, n\pi \sin(m\pi\xi) \cos(n\pi\eta)+2s^2RC_{mn}\, mn\pi^2 \cos(m\pi\xi) \cos(n\pi\eta))$$

$$+ (a_{55}-8d_{55}+16f_{55})(s^3A_{mn}\, m\pi \cos(m\pi\xi) \sin(n\pi\eta)-s^2C_{mn}\, m^2\pi^2$$

$$\sin(m\pi\xi)\ \sin(n\pi\eta)) + (\ a_{44}-8d_{44}+16f_{44}\ )(-s^3RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$-s^2R^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)) + N_0k_1C_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$+ N_0R^2k_2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta) + N_0Rk_3C_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)\ )$$

$$- (4/3)f_{11}sA_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta) + (4/3)sRf_{12}$$

$$B_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \sin(n\pi\eta) + (4/3)f_{16}(-sRA_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta)$$

$$-sB_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \cos(n\pi\eta)) - (16/9)h_{11}(-sA_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)$$

$$+C_{mn}\ m^4\pi^4\ \sin(m\pi\xi)\ \sin(n\pi\eta)) - (16/9)h_{12}(sRB_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$+R^2C_{mn}\ m^2n^2\pi^4\ \sin(m\pi\xi)\ \sin(n\pi\eta)) - (16/9)h_{16}(-sRA_{mn}\ m^2n\pi^3$$

$$\sin(m\pi\xi)\ \cos(n\pi\eta)-sB_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \cos(n\pi\eta) - 2RC_{mn}\ m^3n\pi^4$$

$$\cos(m\pi\xi)\ \cos(n\pi\eta)) - (4/3)sR^2f_{12}A_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)$$

$$+ (4/3)sR^3f_{22}B_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\ \sin(n\pi\eta) + (4/3)f_{26}(-sR^3A_{mn}\ n^3\pi^3$$

$$\sin(m\pi\xi)\ \cos(n\pi\eta)-sR^2B_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \cos(n\pi\eta))$$

$$- (16/9)h_{12}(-sR^2A_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)+R^2C_{mn}\ m^2n^2\pi^4$$

$$\sin(m\pi\xi)\ \sin(n\pi\eta)) - (16/9)h_{22}(sR^3B_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$+R^4C_{mn}\ n^4\pi^4\ \sin(m\pi\xi)\ \sin(n\pi\eta)) - (16/9)h_{26}(-sR^3A_{mn}\ n^3\pi^3$$

$$\sin(m\pi\xi)\ \cos(n\pi\eta)-sR^2B_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \cos(n\pi\eta)-2R^3C_{mn}\ mn^3\pi^4$$

$$\cos(m\pi\xi)\ \cos(n\pi\eta)) - (8/3)sRf_{16}A_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta)$$

$$- (8/3)sR^2f_{26}B_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \cos(n\pi\eta) - (8/3)f_{66}(sR^2A_{mn}\ mn^2\pi^3$$

$$\cos(m\pi\xi)\ \sin(n\pi\eta)+sRB_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \sin(n\pi\eta))$$

$$- (32/9)h_{16}(-sRA_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta)-RC_{mn}\ m^3n\pi^4\ \cos(m\pi\xi)\ \cos(n\pi\eta))$$

$$- (32/9)h_{26}(-sR^2B_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \cos(n\pi\eta)-R^3C_{mn}\ mn^3\pi^4$$

$$\cos(m\pi\xi)\ \cos(n\pi\eta)) - (32/9)h_{66}(-sR^2A_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)$$

$$+sRB_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \sin(n\pi\eta)+2R^2C_{mn}\ m^2n^2\pi^4\ \sin(m\pi\xi)\ \sin(n\pi\eta))$$

$$+ \omega^2\ (\ s^2C_{mn}\ \sin(m\pi\xi)\ \sin(n\pi\eta)+16/4032C_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$$

$$+16/4032R^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)+4/315sA_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi\eta)$$

$$-4/315sRB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi\eta)\ )\ \Big\}\ \Big\{\ \sin(p\pi\xi)\ \sin(q\pi\eta)\ \Big\}\ d\xi d\eta = 0$$

(131)

where by inspection, the boundary terms are zero.

We now multiply through by $\sin(p\pi\xi)\sin(q\pi\eta)$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1\int_0^1\Big\{(a_{45}-8d_{45}+16f_{45})(s^3B_{mn}\,m\pi\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)$$
$$\sin(q\pi\eta)+s^3RA_{mn}\,n\pi\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)+2s^2RC_{mn}\,mn\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta))+(a_{55}-8d_{55}+16f_{55})$$
$$(s^3A_{mn}\,m\pi\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)-s^2C_{mn}\,m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta))+(a_{44}-8d_{44}+16f_{44})(-s^3RB_{mn}\,n\pi\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)-s^2R^2C_{mn}\,n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta))$$
$$+N_0k_1C_{mn}\,m^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)+N_0R^2k_2C_{mn}\,n^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)+N_0Rk_3C_{mn}\,mn\pi^2\cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)\Big)-(4/3)f_{11}sA_{mn}\,m^3\pi^3\cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)+(4/3)sRf_{12}B_{mn}\,m^2n\pi^3\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)$$
$$+(4/3)f_{16}(-sRA_{mn}\,m^2n\pi^3\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$
$$-sB_{mn}\,m^3\pi^3\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta))-(16/9)h_{11}(-sA_{mn}\,m^3\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)+C_{mn}\,m^4\pi^4\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta))-(16/9)h_{12}(sRB_{mn}\,m^2n\pi^3\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)+R^2C_{mn}\,m^2n^2\pi^4\sin(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta))$$
$$-(16/9)h_{16}(-sRA_{mn}\,m^2n\pi^3\sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)$$
$$-sB_{mn}\,m^3\pi^3\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta)-2RC_{mn}\,m^3n\pi^4$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta))-(4/3)sR^2f_{12}A_{mn}\,mn^2\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)+(4/3)sR^3f_{22}B_{mn}\,n^3\pi^3\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)+(4/3)f_{26}(-sR^3A_{mn}\,n^3\pi^3\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)-sR^2B_{mn}\,mn^2\pi^3\cos(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\sin(q\pi\eta))$$
$$-(16/9)h_{12}(-sR^2A_{mn}\,mn^2\pi^3\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)+R^2C_{mn}\,m^2n^2\pi^4$$

$$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta))\ -\ (16/9)h_{22}(sR^3B_{mn}\ n^3\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+R^4C_{mn}\ n^4\pi^4\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta))\ -\ (16/9)h_{26}(-sR^3A_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)-sR^2B_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)$$
$$-2R^3C_{mn}\ mn^3\pi^4\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta))\ -\ (8/3)sRf_{16}A_{mn}\ m^2n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)-\ (8/3)sR^2f_{26}B_{mn}\ mn^2\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)\ +\ (8/3)f_{66}(-sR^2A_{mn}\ mn^2\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+sRB_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta))\ -\ (32/9)h_{16}(-sRA_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)-RC_{mn}\ m^3n\pi^4\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)$$
$$-\ (32/9)h_{26}(-sR^2B_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta)-R^3C_{mn}\ mn^3\pi^4$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi\eta)\sin(q\pi\eta))\ -\ (32/9)h_{66}(-sR^2A_{mn}\ mn^2\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+sRB_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)+2R^2C_{mn}\ m^2n^2\pi^4\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)$$
$$+\ \omega^2\ (\ s^2C_{mn}\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+16/4032C_{mn}\ m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)+16/4032R^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)+4/315sA_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)$$
$$-4/315sRB_{mn}\ n\pi\ \sin(m\pi\xi)\sin(p\pi\xi)\ \sin(n\pi\eta)\sin(q\pi\eta)\ )\ \Big\}\ d\xi d\eta\ =\ 0 \qquad (132)$$

We now simplify the integration as before

$$\sum_{m=1}^{\infty}\ \sum_{n=1}^{\infty}\ \Big\{\ \Big[\ (0\text{ or }0\text{ even},\ 2q/\pi(q^2-n^2)\text{ odd})(1/2\text{ or }0)(s^3m\pi)$$
$$(a_{55}-8d_{55}+16f_{55})\ +\ (0\text{ or }0\text{ even},\ 2p/\pi(p^2-m^2)\text{ odd})(1/2\text{ or }0)$$
$$(sm^3\pi^3)(-4/3f_{11}+16/9h_{11})\ +\ (0\text{ or }0\text{ even},\ 2p/\pi(p^2-m^2)\text{ odd})(1/2\text{ or }0)$$
$$(sR^2mn^2\pi^3)(-4/3f_{12}+16/9h_{12}-8/3f_{66}+32/9h_{66})\ +\ (0\text{ or }0\text{ even},$$

$2p/\pi(p^2-m^2)$ odd)(1/2 or 0)($4/315\omega^2 sm\pi$) + (1/2 or 0)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)($s^3Rn\pi$)($a_{45}-8d_{45}+16f_{45}$) + (1/2 or 0)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)($sRm^2n\pi^3$)($-4f_{16}+48/9h_{16}$) + (1/2 or 0)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)($sR^3n^3\pi^3$)($-4/3f_{26}+16/9h_{26}$) $\Big]$ A_{mn}

+ $\Big[$ (1/2 or 0)(1/2 or 0)($s^3Rn\pi$)($-a_{44}+8d_{44}-16f_{44}$) + (1/2 or 0)(1/2 or 0)

($sRm^2n\pi^3$)($4/3f_{12}-16/9h_{12}+8/3f_{66}-32/9h_{26}$) + (1/2 or 0)(1/2 or 0)

($sR^3n^3\pi^3$)($4/3f_{22}-16/9h_{22}$) + (1/2 or 0)(1/2 or 0)($-4/315\bar{\omega}^2 sRn\pi$)

(0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)($sm\pi$)

($a_{45}-8d_{45}+16f_{45}$) + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)($sm^3\pi^3$)($16/9h_{16}$) + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)

(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)($sR^2mn^2\pi$)($-4f_{26}+48/9h_{26}$) $\Big]$ B_{mn}

+ $\Big[$ (1/2 or 0)(1/2 or 0)($s^2m^2\pi^2$)($-a_{55}+8d_{55}-16f_{55}$) + (1/2 or 0)

(1/2 or 0)($s^2R^2n^2\pi^2$)($-a_{44}+8d_{44}-16f_{44}$) + (1/2 or 0)(1/2 or 0)($\bar{N}_0k_1m^2\pi^2$

$+\bar{N}_0R^2k_2n^2\pi^2+\bar{\omega}^2s^2+16/4032\bar{\omega}^2m^2\pi^2+16/4032\bar{\omega}^2R^2n^2\pi^2$) + (1/2 or 0)(1/2 or 0)

(π^4)($-16/9h_{11}m^4-32/9h_{12}R^2m^2n^2-16/9h_{22}R^4n^4-64/9h_{66}R^2m^2n^2$) +

(0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)($2s^2Rmn\pi^2$)

($a_{45}-8d_{45}+16f_{45}$) + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)(0 or 0 even,

$2q/\pi(q^2-n^2)$ odd)($\bar{N}_0Rk_3mn\pi^2$) + (0 or 0 even, $2p/\pi(p^2-m^2)$ odd)

(0 or 0 even, $2q/\pi(q^2-n^2)$ odd)($64/9\pi^4$)($h_{16}Rm^3n+h_{26}R^3mn^3$) $\Big]$ C_{mn} $\Big\}$ = 0

(133)

These equations will be programmed as before.

For the $\delta\psi_x$ equation of motion we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\!\!\int_0^1 \Big\{ -d_{11}s^2A_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta) - d_{12}s^2RB_{mn}\ mn\pi^2$$

$$\cos(m\pi\xi)\sin(n\pi\eta) + d_{16}(s^2RA_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$-s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)) - (4/3)f_{11}(-s^2A_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$-sC_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) - (4/3)f_{12}(-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)$$

$$-sR^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta)) - (4/3)f_{16}(s^2RA_{mn}\ mn\pi^2$$

$$\cos(m\pi\xi)\cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)-2sRC_{mn}\ m^2n\pi^3$$

$$\sin(m\pi\xi)\cos(n\pi\eta)) + d_{16}s^2RA_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$- d_{26}s^2R^2B_{mn}\ n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta) + d_{66}(-s^2R^2A_{mn}\ n^2\pi^2$$

$$\sin(m\pi\xi)\sin(n\pi\eta)-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)) - (4/3)f_{16}(s^2RA_{mn}$$

$$mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)-sRC_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta))$$

$$- (4/3)f_{26}(-s^2R^2B_{mn}\ n^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)-sR^3C_{mn}\ n^3\pi^3$$

$$\sin(m\pi\xi)\cos(n\pi\eta)) - (4/3)f_{66}(-s^2R^2A_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)-2sR^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(n\pi\eta))$$

$$- (\ a_{45}-8d_{45}+16f_{45}\ )(s^4B_{mn} \sin(m\pi\xi)\cos(n\pi\eta)+s^3RC_{mn}\ n\pi$$

$$\sin(m\pi\xi)\cos(n\pi\eta)) - (\ a_{55}-8d_{55}+16f_{55}\ )(s^4A_{mn} \sin(m\pi\xi)\sin(n\pi\eta)$$

$$+s^3C_{mn}\ m\pi \cos(m\pi\xi)\sin(n\pi\eta)) + (4/3)f_{11}s^2A_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$+ (4/3)\ f_{12}s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta) - (4/3)f_{16}(s^2RA_{mn}\ mn\pi^2$$

$$\cos(m\pi\xi)\cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)) + (16/9)h_{11}$$

$$(-s^2A_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)-sC_{mn}\ m^3\pi^3 \cos(m\pi\xi)\sin(n\pi\eta))$$

$$+ (16/9)h_{12}(-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)-sR^2C_{mn}\ mn^2\pi^3$$

$$\cos(m\pi\xi)\sin(n\pi\eta)) + (16/9)h_{16}(s^2RA_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta)$$

$$-s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\cos(n\pi\eta)-2sRC_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta))$$

$$- (4/3)f_{16}s^2RA_{mn}\ mn\pi^2 \cos(m\pi\xi)\cos(n\pi\eta) + (4/3)f_{26}s^2R^2B_{mn}\ n^2\pi^2$$

$$\sin(m\pi\xi)\cos(n\pi\eta) + (4/3)f_{66}(s^2R^2A_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(n\pi\eta)$$

$$-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(n\pi\eta)) - (16/9)h_{16}(-s^2RA_{mn}\ mn\pi^2$$

$$\cos(m\pi\xi)\cos(n\pi\eta)-sRC_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\cos(n\pi\eta)) + (16/9)h_{26}(-s^2R^2B_{mn}$$

$$n^2\pi^2 \sin(m\pi\xi) \cos(n\pi\eta) - sR^3 C_{mn}\ n^3\pi^3 \sin(m\pi\xi) \cos(n\pi\eta)) + (16/9)h_{66}$$
$$(-s^2R^2A_{mn}\ n^2\pi^2 \sin(m\pi\xi) \sin(n\pi\eta) - s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi) \sin(n\pi\eta)$$
$$-2sR^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi) \sin(n\pi\eta)) + 17/315\omega^2 s^2 A_{mn} \sin(m\pi\xi) \sin(n\pi\eta)$$
$$- 4/315\omega^2 sC_{mn}\ m\pi \cos(m\pi\xi) \sin(n\pi\eta) \Big\} \Big\{ \sin(p\pi\xi)\sin(q\pi\eta) \Big\}\, d\xi d\eta = 0 \tag{134}$$

where the boundaries are zero.

We now simplify the previous equation by multiplying through with $\{\sin(p\pi\xi)\sin(q\pi\eta)\}$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1\!\!\int_0^1 \Big\{ -d_{11}s^2A_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - d_{12}$$
$$s^2RB_{mn} mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) + d_{16}(s^2RA_{mn}\ mn\pi^2 \sin(m\pi\xi)$$
$$\cos(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) - s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (4/3)f_{11}(-s^2A_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - sC_{mn}\ m^3\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{12}(-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - sR^2C_{mn}\ mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}(s^2RA_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta) - s^2B_{mn}\ m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) - 2sRC_{mn}$$
$$m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) + d_{16}s^2RA_{mn}\ mn\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) - d_{26}s^2R^2B_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta) + d_{66}(-s^2R^2A_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$
$$-s^2RB_{mn}\ mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}(s^2RA_{mn}$$
$$mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) - sRC_{mn}\ m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{26}(-s^2R^2B_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta) - sR^3C_{mn}\ n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (4/3)f_{66}(-s^2R^2A_{mn}\ n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$

$$-s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-2sR^2C_{mn}\, mn^2\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi)\sin(n\pi\eta)\sin(q\pi\eta)) - (\ a_{45}-8d_{45}+16f_{45}\ )(s^4B_{mn} \sin(m\pi\xi)$$
$$\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)+s^3RC_{mn}\, n\pi \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$
$$- (\ a_{55}-8d_{55}+16f_{55}\ )(s^4A_{mn} \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$
$$+s^3C_{mn}\, m\pi \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (4/3)f_{11}s^2A_{mn}\, m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) + (4/3)\ f_{12}s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) - (4/3)f_{16}(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta)-s^2B_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$
$$+ (16/9)h_{11}(-s^2A_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$
$$-sC_{mn}\, m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{12}(-s^2RB_{mn}\, mn\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-sR^2C_{mn}\, mn^2\pi^3\cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{16}(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)$$
$$\sin(q\pi\eta)-s^2B_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-2sRC_{mn}\, m^2n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) - (4/3)f_{16}s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{26}s^2R^2B_{mn}\, n^2\pi^2\sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\sin(q\pi\eta) + (4/3)f_{66}(s^2R^2A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)$$
$$\sin(q\pi\eta)-s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)) - (16/9)h_{16}$$
$$(-s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-sRC_{mn}\, m^2n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)) + (16/9)h_{26}(-s^2R^2B_{mn}n^2\pi^2 \sin(m\pi\xi)$$
$$\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta)-sR^3C_{mn}\, n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\sin(q\pi\eta))$$
$$+ (16/9)h_{66}(-s^2R^2A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-s^2RB_{mn}\, mn^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)-2sR^2C_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\sin(q\pi\eta)) + 17/315\omega^2s^2A_{mn} \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta)$$
$$\left.- 4/315\bar{\omega}^2sC_{mn}\, m\pi \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\sin(q\pi\eta) \right\} d\xi\ d\eta = 0 \qquad (135)$$

Using the integration notation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg\{\Bigg[(1/2\text{or }0)(1/2\text{or }0)(s^2m^2\pi^2)(-d_{11}+8/3f_{11}-16/9h_{11}) + (1/2\text{or }0)$$

$$(1/2\text{ or }0)(s^2R^2n^2\pi^2)(4/3f_{16}-d_{66}+8/3f_{66}-16/9h_{66}) + (1/2\text{ or }0)(1/2\text{ or }0)$$

$$(s^4)(-a_{55}+8d_{55}-16f_{55}) + (1/2\text{ or }0)(1/2\text{ or }0)(17/315\bar{\omega}^2s^2)$$

$$(0\text{ or }0\text{ even, }2p/\pi(p^2-m^2)\text{ odd})(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(s^2Rmn\pi^2)$$

$$(2d_{16}-4/3f_{12}-16/3f_{16}+32/9h_{16})\Bigg]A_{mn}$$

$$+\Bigg[(0\text{ or }0\text{ even, }2p/\pi(p^2-m^2)\text{ odd})(1/2\text{ or }0)(s^2Rmn\pi^2)$$

$$(-d_{12}-d_{66}+8/3f_{66}+4/3f_{12}-16/9h_{12}-16/9h_{66}) + (1/2\text{ or }0)$$

$$(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(s^2m^2\pi^2)(-d_{16}+4/3f_{16}-16/9h_{16}) + (1/2\text{ or }0)$$

$$(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})\ (s^4)\ (-a_{45}+8d_{45}-16f_{45}) + (1/2\text{ or }0)$$

$$(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(s^2R^2n^2\pi^2)(-d_{26}+8/3f_{26}-16/9h_{26})\Bigg]B_{mn}$$

$$+\Bigg[(0\text{ or }0\text{ even, }2p/\pi(p^2-m^2)\text{ odd})(1/2\text{ or }0)(sR^2mn^2\pi^3)$$

$$(4/3f_{12}+8/3f_{66}-16/9h_{12}-32/9h_{66}) + (0\text{ or }0\text{ even, }2p/\pi(p^2-m^2)\text{ odd})$$

$$(1/2\text{ or }0)(sm^3\pi^3)(-16/9h_{11}) + (0\text{ or }0\text{ even, }2p/\pi(p^2-m^2)\text{ odd})(1/2\text{ or }0)$$

$$(s^3m\pi)(-a_{55}+8d_{55}-16f_{55}) + (0\text{ or }0\text{ even, }2p/\pi(p^2-m^2)\text{ odd})(1/2\text{ or }0)$$

$$(-4/315\bar{\omega}^2sm\pi) + (1/2\text{ or }0)(0\text{ or }0\text{ even, }2q/\pi(q^2-n^2)\text{ odd})(sRm^2n\pi^3)$$

$$(4f_{16}-48/9h_{16}) + (1/2\text{ or }0)(0\text{ or }0\text{ even, }2m/\pi(m^2-p^2)\text{ odd})(sR^3n^3\pi^3)$$

$$(4/3f_{26}-16/9h_{26})\Bigg]C_{mn}\Bigg\} = 0 \qquad (136)$$

For the $\delta\psi_y$ equation of motion we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1\!\!\int_0^1\Big\{d_{12}s^2RA_{mn}\ mn\pi^2\cos(m\pi\xi)\cos(n\pi\eta) - d_{22}s^2R^2B_{mn}\ n^2\pi^2$$

$$\sin(m\pi\xi)\cos(n\pi\eta) + d_{26}(-s^2R^2A_{mn}\ n^2\pi^2\sin(m\pi\xi)\sin(n\pi\eta) - s^2RB_{mn}\ mn\pi^2$$

$\cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{12}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta)) - (4/3)f_{22}(-s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)$

$\cos(n\pi\eta)-sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$-(4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)-s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \sin(n\pi\eta)$

$-2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - d_{16}s^2A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$- d_{26}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \sin(n\pi\eta) + d_{66}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-sB_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \cos(n\pi\eta)) - (4/3)f_{16}(-s^2A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$-sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{26}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$-s^2R^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{66}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)$

$\cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \cos(n\pi\eta)-2s^2RC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$-(\ a_{44}-8d_{44}+16f_{44}\ )(s^4B_{mn}\ \sin(m\pi\xi)\ \cos(n\pi\eta)+s^3RC_{mn}\ n\pi\ \sin(m\pi\xi)$

$\cos(n\pi\eta)) - (\ a_{45}-8d_{45}+16f_{45}\ )(s^4A_{mn}\ \sin(m\pi\xi)\ \sin(n\pi\eta)+s^3C_{mn}\ m\pi$

$\cos(m\pi\xi)\ \sin(n\pi\eta)) - (4/3)f_{12}s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)$

$+ (4/3)f_{22}s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \cos(n\pi\eta) - (4/3)f_{26}(-s^2R^2A_{mn}\ n^2\pi^2$

$\sin(m\pi\xi)\ \sin(n\pi\eta)-s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \sin(n\pi\eta))$

$+ (16/9)h_{12}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)-sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$+ (16/9)h_{22}(-s^2R^2B_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \cos(n\pi\eta)-sR^3C_{mn}\ n^3\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))$

$+ (16/9)h_{26}(-s^2R^2A_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)-s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \sin(n\pi\eta)$

$-2sR^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) + (4/3)f_{16}s^2A_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi\eta)$

$+ (4/3)f_{26}s^2RB_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \sin(n\pi\eta) + (4/3)f_{66}(-s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)$

$\cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \cos(n\pi\eta)) + (16/9)h_{16}(-s^2A_{mn}\ m^2\pi^2$

$\sin(m\pi\xi)\ \sin(n\pi\eta)-sC_{mn}\ m^3\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta)) + (16/9)h_{26}(-s^2RB_{mn}\ mn\pi^2$

$\cos(m\pi\xi)\ \sin(n\pi\eta)-s^2R^2C_{mn}\ mn^2\pi^3\ \cos(m\pi\xi)\ \sin(n\pi\eta))$

$+ (16/9)h_{66}(s^2RA_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi\eta)-s^2B_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \cos(n\pi\eta)$

$-2sRC_{mn}\ m^2n\pi^3\ \sin(m\pi\xi)\ \cos(n\pi\eta))+\ \omega^2\ (\ 17/315s^2B_{mn}\ \sin(m\pi\xi)$

$\cos(n\pi\eta)\ -\ 4/315sC_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi\eta))\ \Big\}$

$$\left\{ \sin(p\pi\xi)\ \cos(q\pi\eta) \right\} d\xi\ d\eta$$

$$+ \sum_{m=1}^{\tau} \sum_{n=1}^{\tau} \int_0^1 \Bigg\{ \Bigg[d_{12}s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(0) - d_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)$$

$$\sin(0) + d_{26}(s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(0) + s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0))$$

$$- (4/3)f_{12}(s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(0) - sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(0))$$

$$- (4/3)f_{22}(-s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(0) - sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(0))$$

$$- (4/3)f_{26}(s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(0) + s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0)$$

$$+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(0)) - (8/3)f_{12}s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(0)$$

$$+ (8/3)f_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(0) - (8/3)f_{26}(s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(0)$$

$$+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0)) + (32/9)h_{12}(s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(0)$$

$$-sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(0)) + (32/9)h_{22}(-s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(0)$$

$$-sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(0)) + (32/9)h_{26}(s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(0)$$

$$+s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(0) + 2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(0)) \Bigg]$$

$$\Big[\sin(p\pi\xi)\ \cos(0) \Big]$$

$$+ \Bigg[-d_{12}s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi) + d_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi)$$

$$+ d_{26}(-s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi) - s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(n\pi))$$

$$- (4/3)f_{12}(-s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi) + sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi))$$

$$- (4/3)f_{22}(s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi) + sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi))$$

$$- (4/3)f_{26}(-s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi) - s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(n\pi)$$

$$-2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\ \cos(n\pi)) + (8/3)f_{12}s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi)$$

$$- (8/3)f_{22}s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi) - (8/3)f_{26}(-s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)$$

$$\cos(n\pi) - s^2B_{mn}\ m\pi\ \cos(m\pi\xi)\ \cos(n\pi)) + (32/9)h_{12}(-s^2A_{mn}\ m\pi\ \cos(m\pi\xi)\ \sin(n\pi)$$

$$+sC_{mn}\ m^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi)) + (32/9)h_{22}(s^2RB_{mn}\ n\pi\ \sin(m\pi\xi)\ \sin(n\pi)$$

$$+sR^2C_{mn}\ n^2\pi^2\ \sin(m\pi\xi)\ \sin(n\pi)) + (32/9)h_{26}(-s^2RA_{mn}\ n\pi\ \sin(m\pi\xi)\ \cos(n\pi)$$

$$
-s^2 B_{mn}\, m\pi \cos(m\pi\xi) \cos(n\pi) + 2sRC_{mn}\, mn\pi^2 \cos(m\pi\xi) \cos(n\pi)) \Big]
$$

$$
\Big[\sin(p\pi\xi) \cos(q\pi) \Big] \Big\} d\xi = 0 \qquad (137)
$$

We now will simplify the above by using $\sin(0) = \sin(m\pi) = 0$ and $\cos(0) = 1$ to yield

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 \Big\{ d_{12} s^2 RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) - d_{22}
$$
$$
s^2R^2B_{mn} n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) + d_{26}(-s^2R^2A_{mn}\, n^2\pi^2
$$
$$
\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) - s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)
$$
$$
\cos(q\pi\eta)) - (4/3)f_{12}(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta)
$$
$$
-sRC_{mn}\, m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{22}(-s^2R^2B_{mn}
$$
$$
n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) - sR^3C_{mn}\, n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)
$$
$$
\cos(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{26}(-s^2R^2A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)
$$
$$
\sin(n\pi\eta)\cos(q\pi\eta) - s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta)
$$
$$
-2sR^2C_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta)) - d_{16}s^2A_{mn}\, m^2\pi^2
$$
$$
\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) - d_{26}s^2RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)
$$
$$
\sin(n\pi\eta)\cos(q\pi\eta) + d_{66}(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta)
$$
$$
-sB_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{16}(-s^2A_{mn}\, m^2\pi^2
$$
$$
\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) - sC_{mn}\, m^3\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)
$$
$$
\sin(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{26}(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)
$$
$$
\cos(q\pi\eta) - s^2R^2C_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) - (4/3)f_{66}
$$
$$
(s^2RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) - s^2B_{mn}\, m^2\pi^2
$$
$$
\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) - 2s^2RC_{mn}\, m^2n\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)
$$
$$
\cos(n\pi\eta)\cos(q\pi\eta)) - (a_{44} - 8d_{44} + 16f_{44})(s^4B_{mn} \sin(m\pi\xi)\sin(p\pi\xi)
$$
$$
\cos(n\pi\eta)\cos(q\pi\eta) + s^3RC_{mn}\, n\pi \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta))
$$

$$- (a_{45} - 8d_{45} + 16f_{45})(s^4 A_{mn} \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) + s^3 C_{mn}\, m\pi$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta)) - (4/3)f_{12} s^2 RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta) + (4/3)f_{22} s^2 R^2 B_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta) - (4/3)f_{26}(-s^2 R^2 A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)$$
$$\cos(q\pi\eta) - s^2 RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta))$$
$$+ (16/9)h_{12}(s^2 RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) - sRC_{mn}\, m^2 n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{22}(-s^2 R^2 B_{mn}\, n^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) - sR^3 C_{mn}\, n^3\pi^3 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{26}(-s^2 R^2 A_{mn}\, n^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\cos(q\pi\eta) - s^2 RB_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta)$$
$$-2sR^2 C_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta)) + (4/3)f_{16} s^2 A_{mn}\, m^2\pi^2$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) + (4/3)f_{26} s^2 RB_{mn}\, mn\pi^2 \cos(m\pi\xi)$$
$$\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) - (4/3)f_{66}(s^2 RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta) - s^2 B_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta)) +$$
$$(16/9)h_{16}(-s^2 A_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) - sC_{mn}\, m^3\pi^3$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{26}(-s^2 RB_{mn}\, mn\pi^2$$
$$\cos(m\pi\xi)\sin(p\pi\xi) \sin(n\pi\eta)\cos(q\pi\eta) - s^2 R^2 C_{mn}\, mn^2\pi^3 \cos(m\pi\xi)\sin(p\pi\xi)$$
$$\sin(n\pi\eta)\cos(q\pi\eta)) + (16/9)h_{66}(s^2 RA_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)$$
$$\cos(q\pi\eta) - s^2 B_{mn}\, m^2\pi^2 \sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta) - 2sRC_{mn}\, m^2 n\pi^3$$
$$\sin(m\pi\xi)\sin(p\pi\xi) \cos(n\pi\eta)\cos(q\pi\eta)) + \omega^2 \,(17/315 s^2 B_{mn} \sin(m\pi\xi)\sin(p\pi\xi)$$
$$\cos(n\pi\eta)\cos(q\pi\eta) - 4/315 sC_{mn} n\pi \sin(m\pi\xi)\sin(p\pi\xi)\cos(n\pi\eta)\cos(q\pi\eta)) \Big\} d\xi d\eta$$
$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 \Big\{ d_{26}(s^2 RA_{mn} n\pi \sin(m\pi\xi)\sin(p\pi\xi) + s^2 B_{mn}\, m\pi \cos(m\pi\xi)\sin(p\pi\xi))$$
$$- (4/3)f_{26}(s^2 RA_{mn}\, n\pi \sin(m\pi\xi)\sin(p\pi\xi) + s^2 B_{mn}\, m\pi \cos(m\pi\xi)\sin(p\pi\xi)$$
$$+2sRC_{mn}\, mn\pi^2 \cos(m\pi\xi)\sin(p\pi\xi)) - (8/3)f_{26}(s^2 RA_{mn}\, n\pi \sin(m\pi\xi)\sin(p\pi\xi)$$
$$+s^2 B_{mn}\, m\pi \cos(m\pi\xi)\sin(p\pi\xi)) + (32/9)h_{26}(s^2 RA_{mn}\, n\pi \sin(m\pi\xi)\sin(p\pi\xi)$$

$$+s^2 B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi))$$

$$+\ d_{26}(-s^2 RA_{mn}\ n\pi\ \sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)-s^2 B_{mn}\ m\pi\ \cos(m\pi\xi)$$

$$\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi))\ -\ (4/3)f_{26}(-s^2 RA_{mn}\ n\pi\ \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi)\cos(q\pi)-s^2 B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)$$

$$-2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi))\ -\ (8/3)f_{26}(-s^2 RA_{mn}\ n\pi$$

$$\sin(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)-s^2 B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi)\cos(q\pi))\ +\ (32/9)h_{26}(-s^2 RA_{mn}\ n\pi\ \sin(m\pi\xi)\sin(p\pi\xi)$$

$$\cos(n\pi)\cos(q\pi)-s^2 B_{mn}\ m\pi\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi)$$

$$+2sRC_{mn}\ mn\pi^2\ \cos(m\pi\xi)\sin(p\pi\xi)\ \cos(n\pi)\cos(q\pi))\ \Big\}\ d\xi = 0 \qquad (138)$$

Using integration notation we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\Big[\ (0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})(1/2 \text{ or } 0)(s^2Rmn\pi^2)$$

$$(d_{12}-8/3f_{12}+d_{66}-4/3f_{26}-4/3f_{66}+16/9h_{12}+16/9h_{66})\ +\ (1/2 \text{ or } 0)$$

$$(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^2R^2n^2\pi^2)(-d_{26}+8/3f_{26}+4/3f_{66}-16/9h_{26})$$

$$+\ (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^2m^2\pi^2)$$

$$(-d_{16}+8/3f_{16}-16/9h_{16})\ +\ (1/2 \text{ or } 0)(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^4)$$

$$(-a_{45}+8d_{45}-16f_{45})\ +\ (1/2 \text{ or } 0)(1-\cos(n\pi)\cos(q\pi))(s^2Rn\pi)$$

$$(d_{26}-4f_{26}+32/9h_{26})\ \Big]\ A_{mn}$$

$$+\ \Big[\ (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^2R^2n^2\pi^2)(-d_{22}+8/3f_{22}-16/9h_{22})\ +\ (1/2 \text{ or } 0)$$

$$(1/2 \text{ or } 0)(s^2m^2\pi^2)(-d_{66}+4/3f_{66}-16/9h_{66})\ +\ (1/2 \text{ or } 0)(1/2 \text{ or } 0)(s^4)$$

$$(-a_{44}+8d_{44}-16f_{44})\ +\ (1/2 \text{ or } 0)(1/2 \text{ or } 0)(17/315\bar{\omega}^2s^2)\ +\ (0 \text{ or } 0 \text{ even},$$

$$2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even},\ 2n/\pi(n^2-q^2) \text{ odd})(s^2Rmn\pi^2)(-2d_{26}+4f_{26}$$

$$-32/9h_{26}+4/3f_{66})\ +\ (0 \text{ or } 0 \text{ even},\ 2p/\pi(p^2-m^2) \text{ odd})(1-\cos(n\pi)\cos(q\pi))$$

$$
\begin{aligned}
&(s^2 m\pi)((d_{26}-4f_{26}+32/9h_{26})\Big]\, B_{mn} \\
&+\Big[\,(1/2 \text{ or } 0)(1/2 \text{ or } 0)(sRm^2 n\pi^3)(4/3f_{12}+8/3f_{66}-16/9h_{12}-32/9h_{66}) \\
&+(1/2 \text{ or } 0)(1/2 \text{ or } 0)(sR^3 n^3\pi^3)(4/3f_{22}-16/9h_{22}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0) \\
&(s^3 Rn\pi)(-a_{44}+8d_{44}-16f_{44}) + (1/2 \text{ or } 0)(1/2 \text{ or } 0)(-4/315\bar{\omega}^2 sn\pi) \\
&(0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even, } 2n/\pi(n^2-q^2) \text{ odd})(sR^2 mn^2\pi^3) \\
&(4f_{26}-48/9h_{26}) + (0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd})(0 \text{ or } 0 \text{ even,} \\
&2n/\pi(n^2-q^2) \text{ odd})(sm^3\pi^3)(4/3f_{16}-16/9h_{16}) + (0 \text{ or } 0 \text{ even, } 2p/\pi(p^2-m^2) \text{ odd}) \\
&(0 \text{ or } 0 \text{ even, } 2n/\pi(n^2-q^2) \text{ odd})(s^3 m\pi)(-a_{45}+8d_{45}-16f_{45}) + (0 \text{ or } 0 \text{ even,} \\
&2p/\pi(p^2-m^2) \text{odd})(1-\cos(n\pi)\cos(q\pi))(sRmn\pi^2)(-8/3f_{26}+64/9h_{26})\Big]\, C_{mn} \Big\} = 0
\end{aligned}
\tag{139}
$$

III. Discussion and Results

We will now focus on the computer routines that were used to accomplish the stated objectives. Discussion will also be given on the two laminated plates chosen, along with all analyses performed on each of the plates.

Computer Routines

Three individual computer programs were written to facilitate solution of the Galerkin equations. Natural frequencies and buckling loads using the equations derived in the last chapter would be very difficult to obtain by hand without the aid of the computer. The first program generates the dimensional and nondimensional plate stiffnesses for a symmetric laminate. The Galerkin algorithm is carried out in the second program. The stiffness and mass/inertia matricies are generated to set up the eigenvalue problem $[A]x=\lambda[B]x$ where $[A]$ is the stiffness matrix and $[B]$ is mass/inertia matrix as seen in Eq.(90) on page 61. A third program was written to solve the eigenvalue problem. Each program will be described below.

The first program generates the dimensional and nondimensional stiffnesses of laminated plate. Only the stiffnesses which correspond to a symmetric plate are given as output. The laminate density is also generated. Each individual lamina requires an orientation angle, distance from midplane, unidirectional material properties, and density as input. Shear moduli G_{13} is set equal to G_{12} and G_{23} is set to 80% of G_{12}. The reduced stiffnesses (Q_{ij}'s) and transformed reduced stiffnesses ($\bar{Q}_{ij}$'s) are then computed. Laminate stiffnesses are next

calculated with the transformed reduced stiffnesses and the position of the lamina with respect to the midplane. Each stiffness is summed within a loop and is continued until each ply has been laid up. Finally, the stiffnesses are printed out, normalized stiffnesses calculated, and normalized stiffnesses printed out. A listing of the program is found in Appendix A.

Program number two generates the stiffness and mass/inertia matricies by carrying out the Galerkin algorithm. This program is unique because it has the ability to give vibrational and buckling solutions for the higher order theory and two first order theories. One first order theory assumes the shear correction coefficient to be 5/6 and the other can use two coefficients of the users choice. The two coefficient option was not exercised for this work.

The program is comprised of basically five sections: input, integration, stiffness matrix generation, mass/inertia matrix generation, and output. The three middle sections are placed within several loops to form all of the terms and equations needed for the Galerkin algorithm.

The first section reads laminate input data from the file "GALIN" and performs minor calculations. Higher order theory, vibration, and rotatory inertia flags are read first to determine the type of problem to be run. For example, if HOT = 1 then the higher order theory will be run; otherwise, if HOT $\neq$ 1 the first order theory will be executed. Next, MMAX and NMAX are read to define the number of terms and equations in the Galerkin algorithm and thus the size of the problem. Plate dimensional data is read next. The following line takes

the plate weight density and transverse modulus along with two shear correction coefficients (for the double correction scheme) and the proportional loading coefficients. Stiffnesses for a symmetric laminate are read next. Higher order stiffnesses are set equal to zero when running the first order theory. Logic to implement either of the first order theory coefficients follows the input section.

We then proceed into the nested loops of the Galerkin algorithm. Integration is performed in this section by using the functions as defined in the last chapter. The terms EOTEST1 and EOTEST2 are logical values to determine whether the sums M+P or N+Q are even or odd respectively.

The third section computes the stiffness terms for each of the three equations of motion. These equations were derived for all three boundary conditions and include the integration functions which are evaluated in the previous section. After all nine values have been computed, they are stored in the matrix [ST]. A negative sign is included to effectively separate the stiffness terms from the mass/inertia terms by moving them to the other side of the equation.

Section four contains the necessary code to compute mass and inertia terms for both vibrational and buckling analyses. It can be seen that the mass/inertia terms for high order theory and the first order theories are quite different for vibrational solutions but are identical for buckling solutions. LAM1 and LAM2 are quantities factored out for convenience.

The last section simply writes the stiffness and mass/inertia matricies to the file "EIGIN" for storage. MMAX is also written to

simplify the eigenvalue solver routine. Complete file listings for "GALERK1", "GALERK2", and "GALERK3" for the three boundary conditions are given in Appendix B.

Program three ,"EIGEN", is the routine used to solve the eigenvalue problem (see Eq.(90) page 61). It simply reads the stiffness and mass/inertia matricies from the file "EIGIN" (which were generated by program number two), calls the IMSL general eigensolver routine GVCRG [17], and prints the results to the file "EIGOUT". The IMSL routine was chosen because it is a ready to run, robust package. The routine GPIRG was is also called from the IMSL package to compute the performance index of the solution. The performance index is a measure of how well GVCRG performed in that it can be used to find the number of significant digits in the solution [18]. A complete program listing may be seen in Appendix C.

Analysis Performed

Characteristics of the first order theory and the higher order theory were investigated for all three boundary conditions. Vibrational and buckling solutions were obtained for all combinations of theories and boundaries. Convergence studies of the Galerkin technique were performed. Span-to-depth ratios were varied to comprehend its effects on the natural frequencies and buckling loads.

Laminated Plate Properties Two different laminated plates were used in the analysis. Plate number one has a layup of $[0_{50}/90_{50}]_s$ and plate number two has a layup of $[\pm 45_{50}]_s$. Both plates were square and one inch thick. The span of each plate was varied depending on the desired span-to-depth ratio. Since the equations of motion are normalized (and therefore the plate stiffnesses are normalized), the number of plys per orientation angle is immaterial. Therefore, for the remainder of this work the $[0_{50}/90_{50}]_s$ plate will designated as $[0/90]_s$ and the $[\pm 45_{50}]_s$ plate designated as $[\pm 45]_s$.

Each plate was comprised of graphite/epoxy material (AS/3501) which has the following unidirectional properties

$$E_1 = 21.0E+06 \text{ psi}$$

$$E_2 = 1.40E+06 \text{ psi}$$

$$\upsilon_{12} = 0.3$$

$$G_{12} = 0.60E+06 \text{ psi}$$

$$\rho = 0.055 \text{ lb./in.}^3$$

$$t = .005 \text{ in.}$$

where ρ is the weight density. The weight density must be converted to a mass density (as shown in program two) in order for consistent units. Tables 3.1 and 3.2 contain the stiffness terms generated by the first program for both plates.

Graphite/Epoxy $[0/90]_s$

Element	Dimensional Value (one inch thick)	Nondimensional Value
A_{44}	540,000	0.3857143
A_{45}	0	0
A_{55}	540,000	0.3857143
D_{44}	41,250	0.0294643
D_{45}	0	0
D_{55}	48,750	0.0348214
F_{44}	6047	0.0043192
F_{45}	0	0
F_{55}	7453	0.0053237
D_{11}	1,555,164	1.110832
D_{12}	35,211	0.0251509
D_{16}	0	0
D_{22}	322,770	0.23055
D_{26}	0	0
D_{66}	50,000	0.0357143
F_{11}	256,382	0.18313
F_{12}	5282	0.0037726
F_{16}	0	0
F_{22}	25,308	0.018077
F_{26}	0	0
F_{66}	0	0
H_{11}	46,814	0.033439
H_{12}	943	0.000673
H_{16}	0	0

H_{22}	3487	0.002491
H_{26}	0	0
H_{66}	1339	0.000957

Units for dimensional A_{ij}, D_{ij}, F_{ij}, and H_{ij} stiffnesses are lb./in., in.-lbs., in.^{3}-lbs., and in.^{5}-lbs. respectively.

Table 3.1 Stiffness Elements for Plate Number 1

Graphite/Epoxy $[\pm 45]_s$

Element	Dimensional Value (one inch thick)	Nondimensional Value
A_{44}	540,000	0.3857143
A_{45}	0	0
A_{55}	540,000	0.3857143
D_{44}	45,000	0.0321429
D_{45}	-3750	-0.0026786
D_{55}	45,000	0.0321429
F_{44}	6750	0.0048214
F_{45}	-703	-0.0005022
F_{55}	6750	0.0048214
D_{11}	537,089	0.3836353
D_{12}	437,089	0.3122066
D_{16}	308,098	0.2200704
D_{22}	537,089	0.3836353
D_{26}	308,098	0.2200704
D_{66}	451,878	0.32277
F_{11}	80,563	0.0575453
F_{12}	65,563	0.046831
F_{16}	57,768	0.0412632
F_{22}	80,563	0.0575453
F_{26}	57,768	0.0412632
F_{66}	67,781	0.0484155
H_{11}	14,386	0.0102759
H_{12}	11,707	0.00836267
H_{16}	10,831	0.00773685

H_{22}	14,386	0.0102759
H_{26}	10,831	0.00773685
H_{66}	12,104	0.00864563

Units for dimensional A_{ij}, D_{ij}, F_{ij}, and H_{ij} stiffnesses are lb./in., in.-lbs., in.^{3}-lbs., and in.^{5}-lbs. respectively.

Table 3.2 Stiffness Elements for Plate Number 2

Galerkin Method Convergence Characteristics Because the Galerkin technique is an approximate solution method, results generated must be investigated to see if they converge to unique values. This convergence as M and N are increased is a necessary but not sufficient condition of the method. Proving sufficient conditions for convergence is very difficult and is beyond the scope of this work. Reference [19] contains complete discussion on the proof of convergence. For this work, we will stipulate that natural frequency and buckling load decrease with increasing M and N.

Table 3.3 shows nondimensional natural frequencies and corresponding values of M and N for the three boundary conditions on the $[0/90]_s$ plate. The simply supported condition implies simple supports on all four sides while the clamped implies clamped edges for all four sides. The clamped-simple boundary condition assumes clamped edges on two opposite sides ($x = 0$ and $x = a$) and simple supports on the two remaining sides ($y = 0$ and $y = b$).

Results include the first order shear deformation theory (using a shear correction coefficient of $k=5/6$) and the higher order shear deformation theory. A span-to-depth ratio of $S=10$ was used for the first order theory while $S=20$ was used for the higher order theory and an aspect ratio of $a/b=1$ (square plate) was chosen for these studies. The plate thickness was one inch. Rotatory inertia was included in the first order theory natural frequency calculations. Figures 3.1 and 3.2 graphically show the results of Table 3.3. The letter C denotes the clamped boundary condition, the letters C-S denote the clamped-simply suppoted boundary condition, and the letter S denotes the simply

supported boundary condition. Nondimensional buckling loads for the $[0/90]_s$ plate are given on Table 3.4 and Figures 3.3 and 3.4.

Corresponding results for the $[\pm 45]_s$ plate are shown on Tables 3.5 and 3.6 and Figures 3.5, 3.6, 3.7, and 3.8.

Nondimensional Natural Frequency $\bar{\omega}_n = \omega_n a^2 (p/E_2 h^3)^{1/2}$

First Order Theory (s=10)

Simple B.C. M	N	$\bar{\omega}_n$	Clamped B.C. M	N	$\bar{\omega}_n$	Clamped-Simple M	N	$\bar{\omega}_n$
2	2	10.64	2	2	19.03	2	2	15.69
4	4	10.64	4	4	17.40	4	4	15.25
6	6	10.64	6	6	17.21	6	6	15.20
8	8	10.64	8	8	17.16	8	8	15.18
10	10	10.64	10	10	17.14	10	10	15.18

High Order Theory (s=20)

Simple B.C. M	N	$\bar{\omega}_n$	Clamped B.C. M	N	$\bar{\omega}_n$	Clamped-Simple M	N	$\bar{\omega}_n$
2	2	11.75	2	2	26.66	2	2	20.96
4	4	11.75	4	4	21.95	4	4	19.52
6	6	11.75	6	6	21.58	6	6	19.44
8	8	11.75	8	8	21.52	8	8	19.46
10	10	11.75	10	10	21.52	10	10	19.50

Table 3.3 Nondimensional Natural Frequency Convergence Characteristics for the $[0/90]_s$ Plate

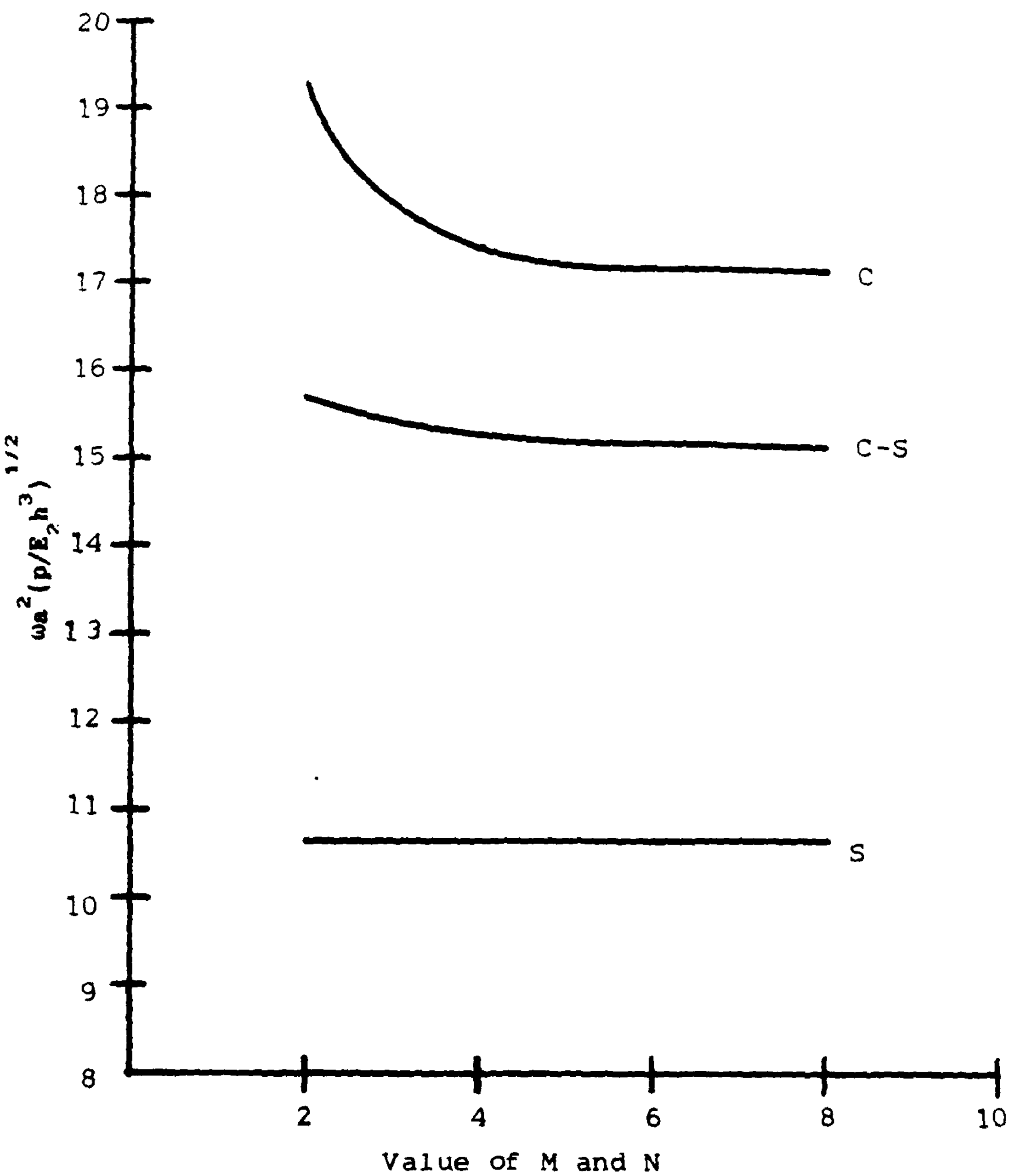

Figure 3.1 Nondimensional Natural Frequency Convergence Characteristics for the $[0/90]_s$ Plate

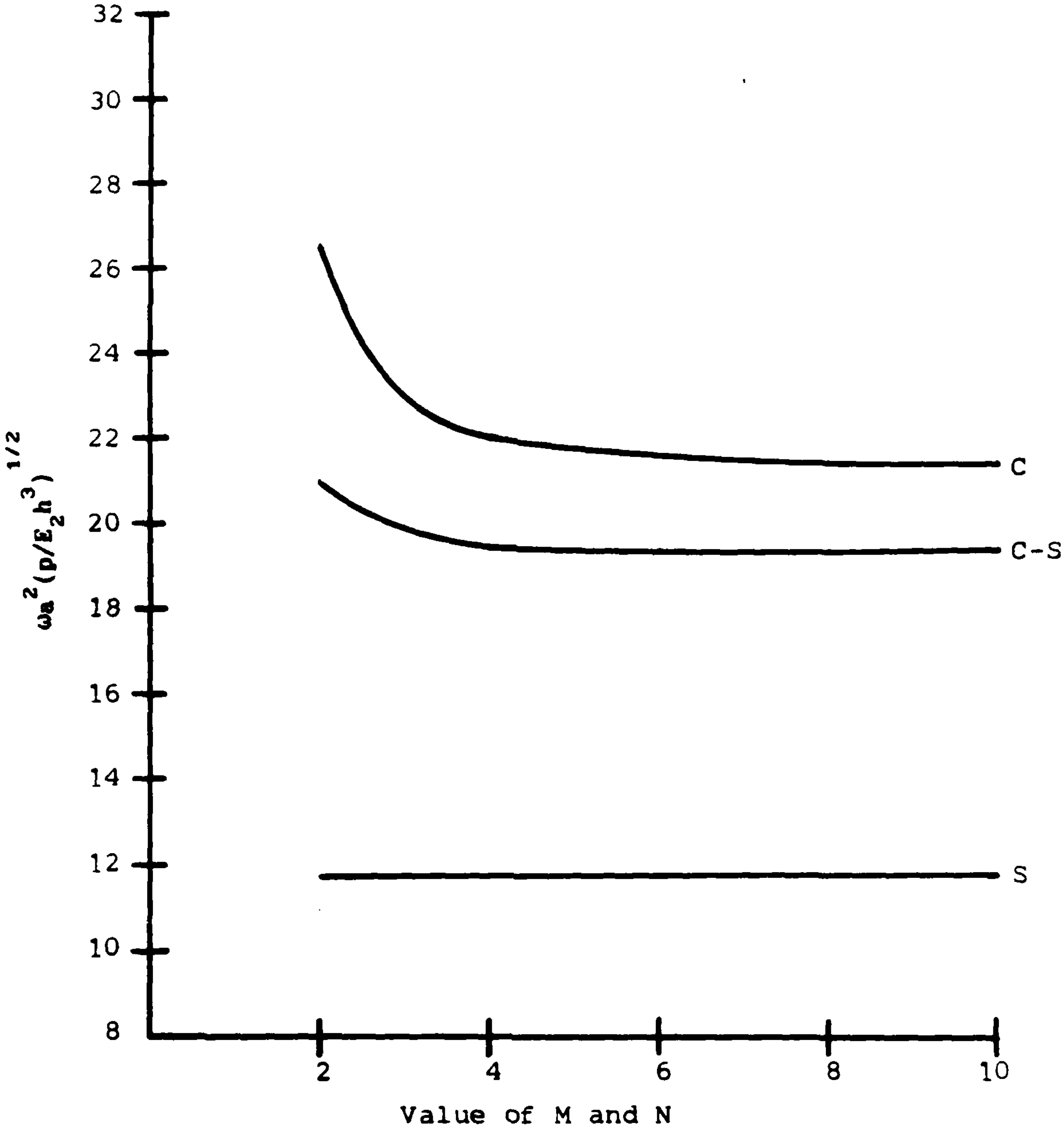

Figure 3.2 Nondimensional Natural Frequency Convergence Characteristics for the $[0/90]_s$ Plate

Nondimensional Buckling Load $\bar{N}_x = N_x a^2 / E_2 h^3$

First Order Theory (s=10)

Simple B.C.			Clamped B.C.			Clamped-Simple		
M	N	$\bar{N}_x$	M	N	$\bar{N}_x$	M	N	$\bar{N}_x$
2	2	11.61	2	2	32.89	2	2	25.16
4	4	11.61	4	4	24.60	4	4	21.06
6	6	11.61	6	6	24.35	6	6	20.92
8	8	11.61	8	8	24.28	8	8	20.89
10	10	11.61	10	10	24.25	10	10	20.88

High Order Theory (s=20)

Simple B.C.			Clamped B.C.			Clamped-Simple		
M	N	$\bar{N}_x$	M	N	$\bar{N}_x$	M	N	$\bar{N}_x$
2	2	14.03	2	2	72.21	2	2	44.65
4	4	14.03	4	4	42.93	4	4	34.89
6	6	14.03	6	6	41.71	6	6	34.62
8	8	14.03	8	8	41.46	8	8	34.66
10	10	14.03	10	10	41.40	10	10	34.73

Table 3.4 Nondimensional Buckling Load Convergence Characteristics for the $[0/90]_s$ Plate

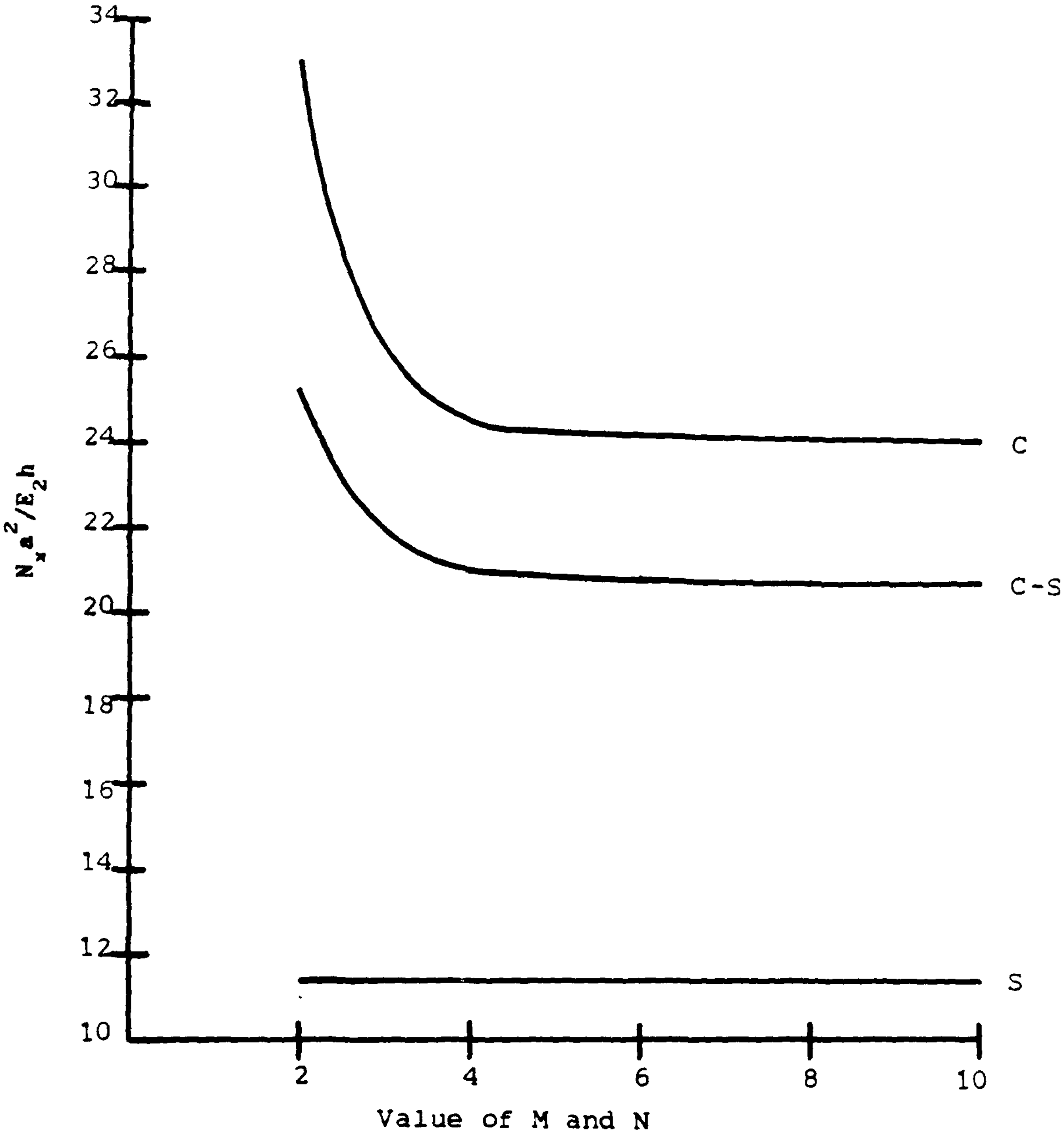

Figure 3.3 Nondimensional Buckling Load Convergence Characteristics for the $[0/90]_s$ Plate

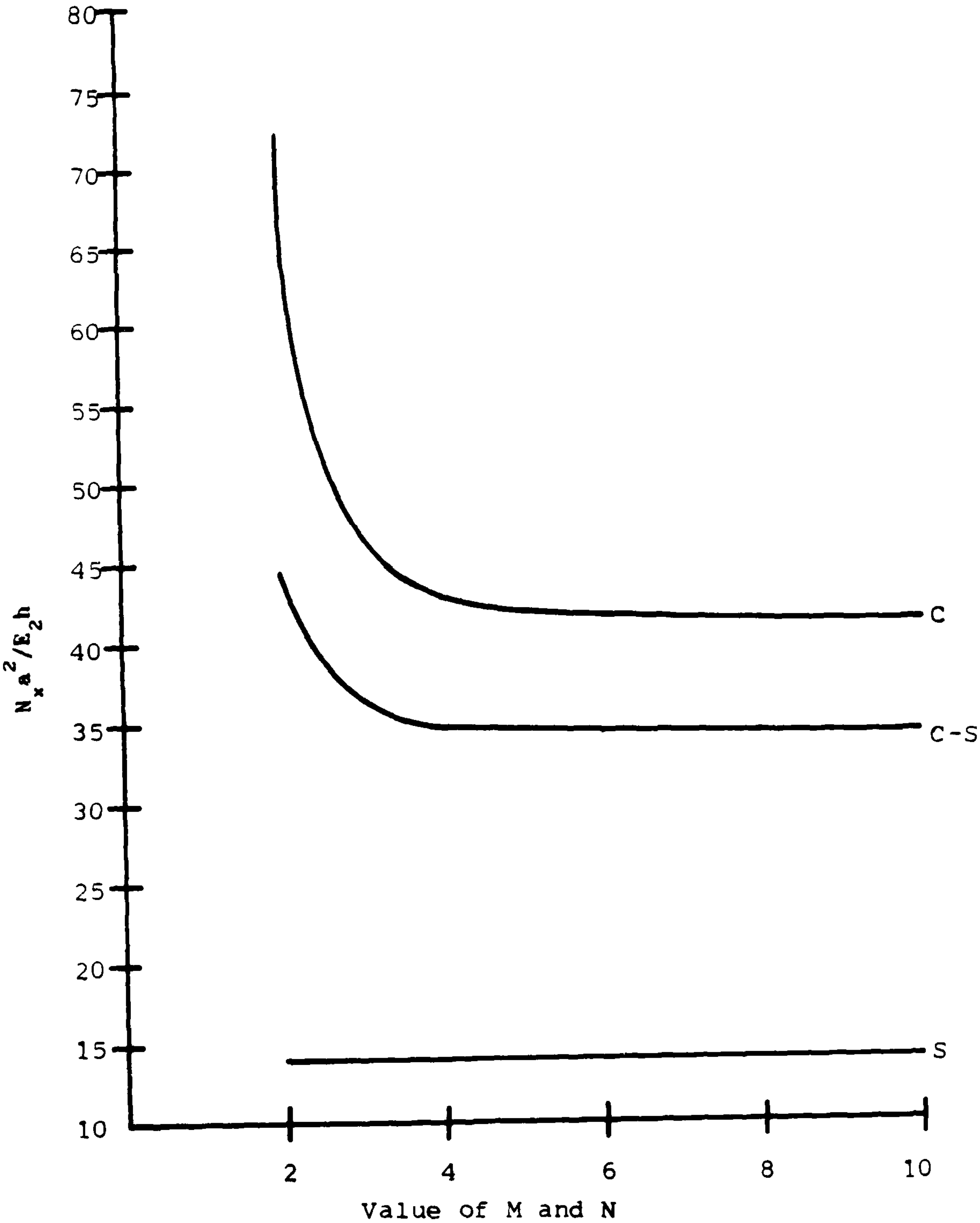

Figure 3.4 Nondimensional Buckling Load Convergence Characteristics for the $[0/90]_s$ Plate

Nondimensional Natural Frequency $\bar{\omega}_n = \omega_n a^2 (p/E_2 h^3)^{1/2}$

First Order Theory (s=10)

Simple B.C.			Clamped B.C.			Clamped-Simple		
M	N	$\bar{\omega}_n$	M	N	$\bar{\omega}_n$	M	N	$\bar{\omega}_n$
2	2	13.09	2	2	18.53	2	2	15.97
4	4	12.88	4	4	16.70	4	4	15.02
6	6	12.78	6	6	16.45	6	6	14.85
8	8	12.72	8	8	16.37	8	8	14.80
10	10	12.68	10	10	16.34	10	10	14.77

High Order Theory (s=20)

Simple B.C.			Clamped B.C.			Clamped-Simple		
M	N	$\bar{\omega}_n$	M	N	$\bar{\omega}_n$	M	N	$\bar{\omega}_n$
2	2	14.56	2	2	26.33	2	2	21.22
4	4	14.28	4	4	20.99	4	4	18.01
6	6	14.17	6	6	20.36	6	6	17.58
8	8	14.11	8	8	20.20	8	8	17.46
10	10	14.06	10	10	20.15	10	10	17.41

Table 3.5 Nondimensional Natural Frequency Convergence Characteristics for the $[\pm 45]_s$ Plate

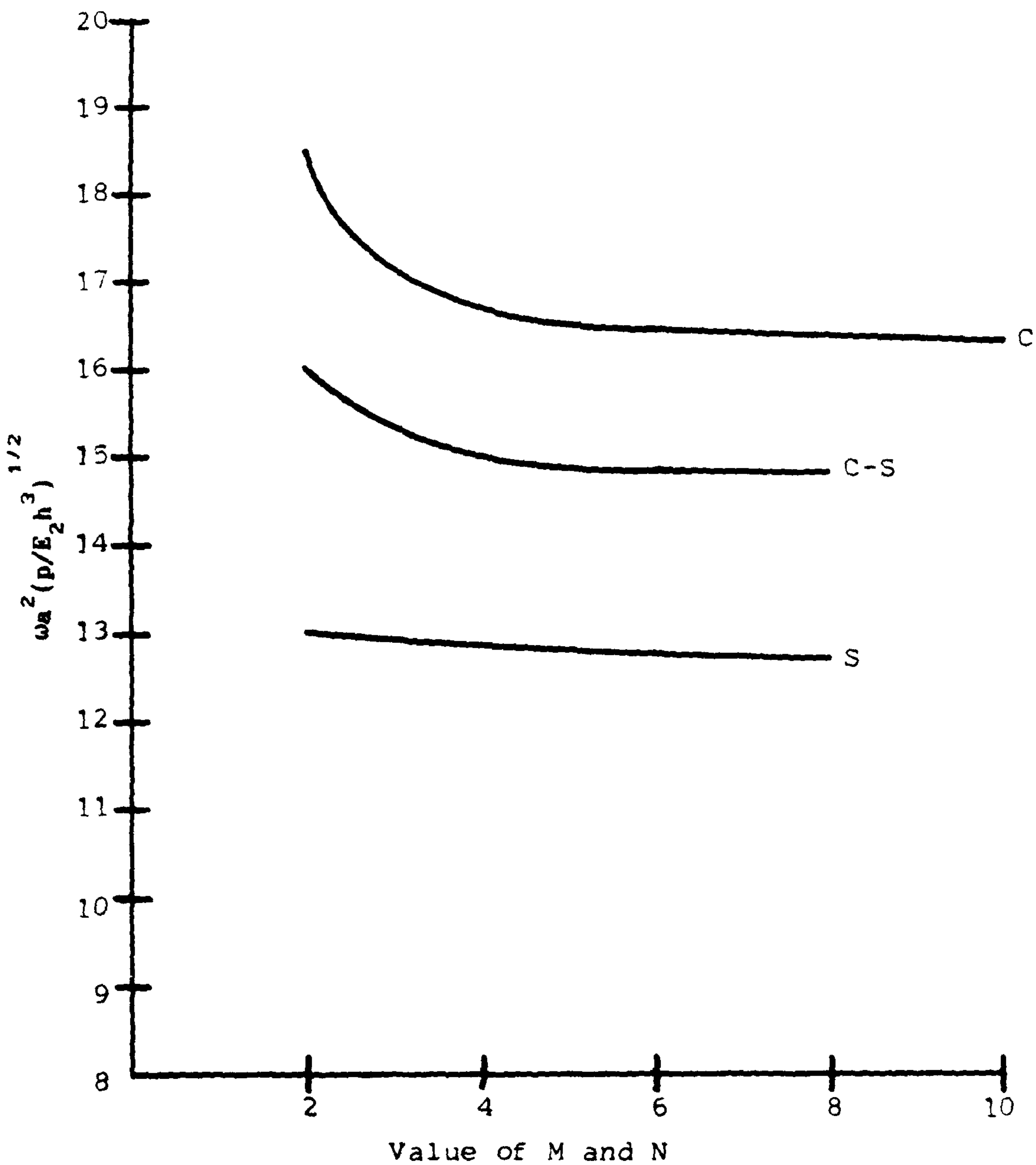

Figure 3.5 Nondimensional Natural Frequency Convergence Characteristics for the $[\pm 45]_s$ Plate

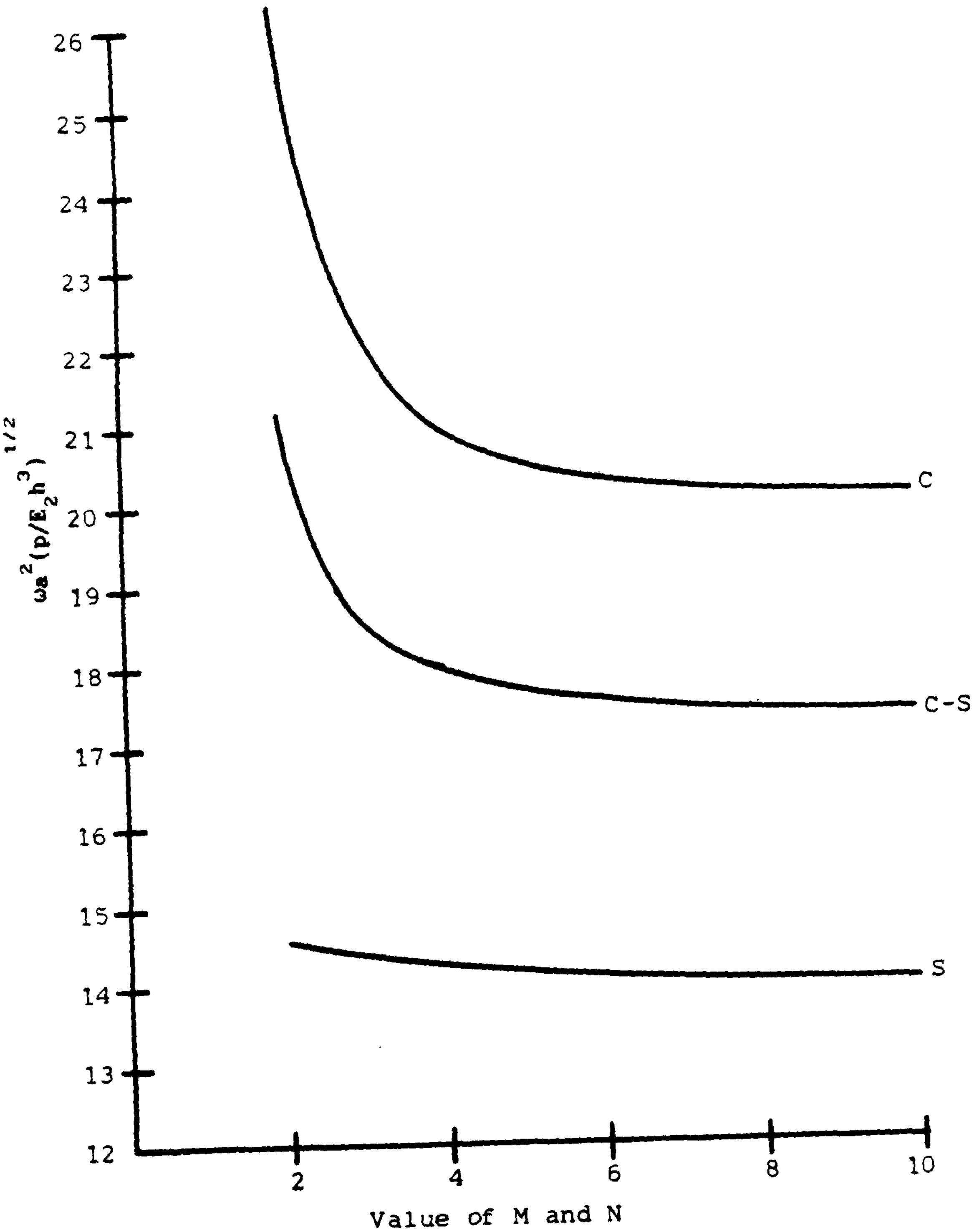

Figure 3.6 Nondimensional Natural Frequency Convergence Characteristics for the $[\pm45]_s$ Plate

Nondimensional Buckling Load $\bar{N}_x = N_x a^2 / E_2 h^3$

First Order Theory (s=10)

Simple B.C.			Clamped B.C.			Clamped-Simple		
M	N	$\bar{N}_x$	M	N	$\bar{N}_x$	M	N	$\bar{N}_x$
2	2	17.21	2	2	32.39	2	2	25.58
4	4	16.31	4	4	18.70	4	4	17.82
6	6	16.03	6	6	18.30	6	6	17.61
8	8	15.91	8	8	18.22	8	8	17.52
10	10	15.81	10	10	18.20	10	10	17.47

High Order Theory (s=20)

Simple B.C.			Clamped B.C.			Clamped-Simple		
M	N	$\bar{N}_x$	M	N	$\bar{N}_x$	M	N	$\bar{N}_x$
2	2	21.13	2	2	69.93	2	2	45.16
4	4	19.86	4	4	29.71	4	4	24.40
6	6	19.55	6	6	28.83	6	6	23.84
8	8	19.31	8	8	28.64	8	8	23.65
10	10	19.19	10	10	28.58	10	10	23.56

Table 3.6 Nondimensional Buckling Load Convergence Characteristics for the $[\pm 45]_s$ Plate

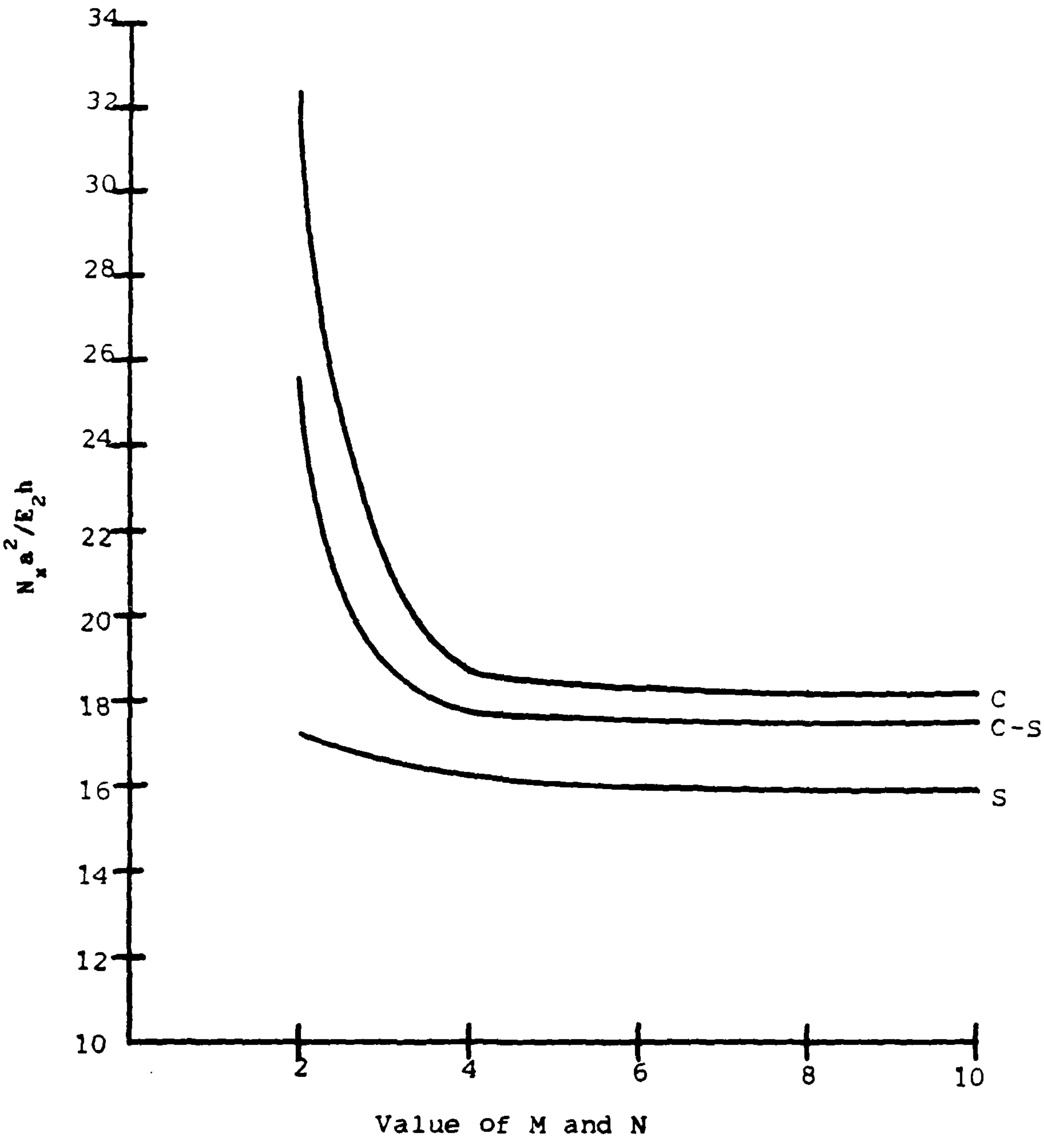

Figure 3.7 Nondimensional Buckling Load Convergence Characteristics for the $[\pm 45]_s$ Plate

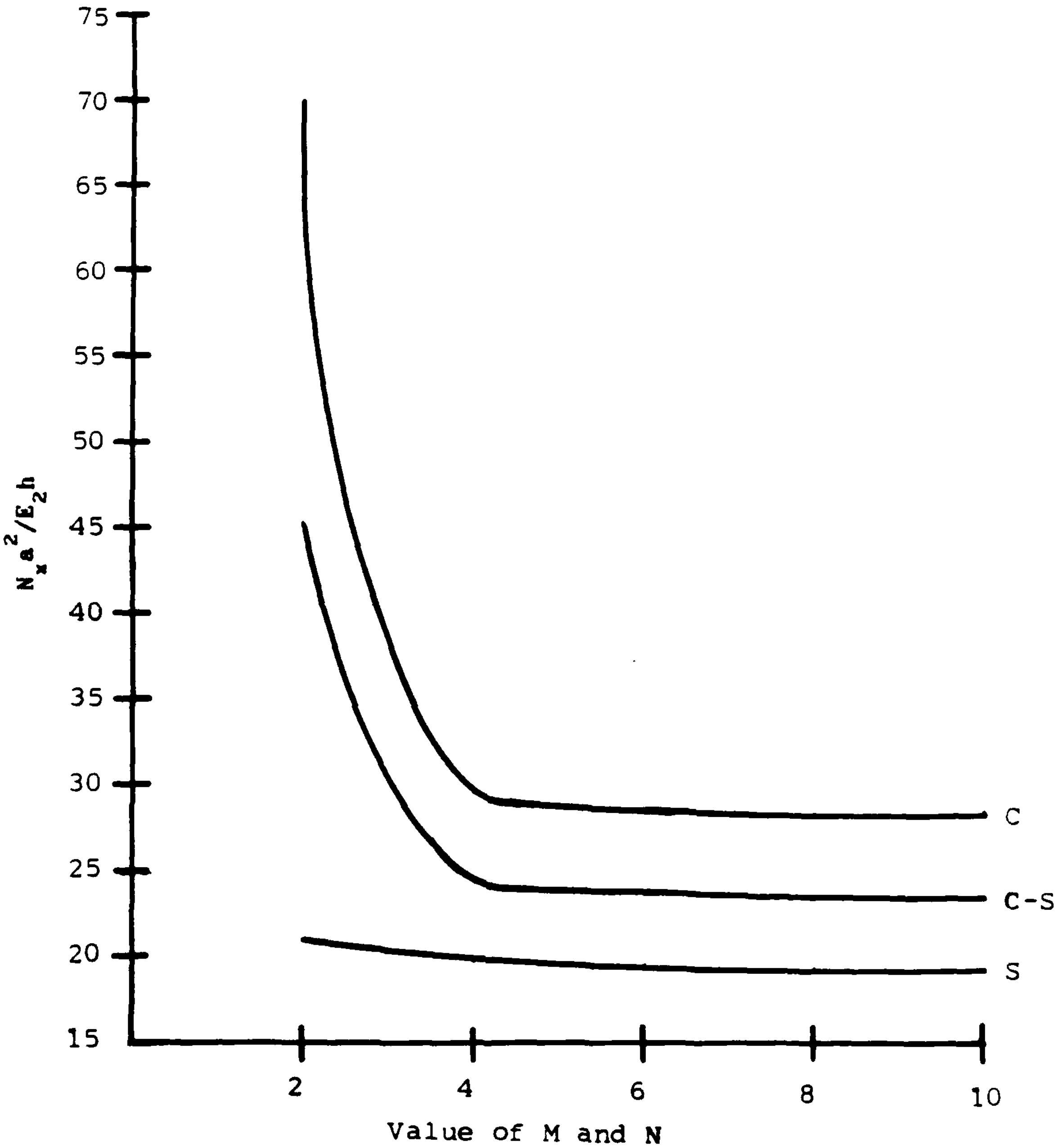

Figure 3.8 Nondimensional Buckling Load Convergence Characteristics for the $[\pm 45]_s$ Plate

Some general observations can be drawn from these results. The first order natural frequencies for the $[0/90]_s$ plate converge nicely and are well behaved. The simply supported values converge at the beginning with M and N equal to 2. This is due to the fact that the natural boundary conditions are satisfied in addition to the geometric boundary conditions. The clamped and clamped-simple boundary cases do not initially converge like the simply supported case but do show good characteristics.

The higher order natural frequencies for the $[0/90]_s$ plate also behave nicely. The simply supported boundary case initially converges as in the first order theory. The clamped boundary also converges well. A difference of only .2% is seen between values for M=N=6 and M=N=8. The clamped-simple case converges, but then appears to diverge. This anomaly was determined to be a loss of significant digits in the solution. Using the performance index from each run and the number of single precision significant digits in the computer, it was found that the answers for M and N greater than 6 had only four significant digits. A change in the computer code to perform double precision arithmetic would have probably shown better convergence characteristics for the clamped-simply supported boundary condition. A difference of .1% is evident between M=N=6 and M=N=8. Values for clamped and clamped-simple are reasonably converged at M=N=4.

Nondimensional buckling loads for the $[0/90]_s$ plate display similar characteristics as the natural frequencies. The first order convergence is good. The higher order convergence is also good with the exception of the clamped-simple boundary at M=N=6 and M=N=8. A loss of

significant digits, as in the natural frequency calculations, was also noted here. More terms are needed for the buckling solutions than in the natural frequency solutions. The difference between M=N=2 and M=N=8 for clamped buckling loads is 42.6% but is only 19.3% for clamped natural frequencies. Differences of 22.4% and 7.2% are seen for the clamped-simple boundary condition in buckling and vibration respectively. Solutions for clamped and clamped-simple appear to be converged at M=N=6 as opposed to M=N=4 for natural frequencies. This tendency in buckling is probably due to the fact that the plate has been assumed to be inextensible (i.e., no u or v displacements). Values between M=N=6 and M=N=8 for clamped and clamped-simple are .6% and .1% respectively.

Results for the $[\pm 45]_s$ plate are similar to the $[0/90]_s$ plate but do show differences. The natural frequencies for both theories with the simply supported boundaries do not converge from the start as did the $[0/90]_s$ plate, but do converge rapidly. This laminate does not satisfy the natural boundary conditions for simple supports. The other two boundaries converge nicely from above for both theories. None of the behavior observed using the higher order theory on the clamped-simple boundaries was seen for the $[\pm 45]_s$ plate. Convergence was a little slower when compared with the $[0/90]_s$ plate. Differences of 23.3% and 17.7% were observed between M=N=2 and M=N=8 for the clamped and clamped-simple boundaries respectively (19.3% and 7.2% was seen for the $[0/90]_s$ plate). All values are reasonably converged at M=N=6 with differences between M=N=6 and M=N=8 of .4%, .8%, and .7% seen for the simple, clamped, and clamped-simple boundary conditions respectively.

The [±45] plate buckling loads results appear similar to the natural frequency results in terms of convergence characteristics. Convergence was again slower than the $[0/90]_s$ plate as first seen for the natural frequencies. Differences between M=N=2 and M=N=8 was 8.6% for the simple supports, 59.0% for the clamped supports, and 47.6% for the clamped-simple supports (differences of 0%, 42.6%, and 22.4% were calculated for the $[0/90]_s$ plate). Convergence at M=N=6 is not quite as good as the other configurations with a 1.2% difference for simple, .6% difference for clamped, and .8% difference for clamped-simple between M=N=6 and M=N=8.

High Order Shear Deformation Effects This area of study is obviously the main thrust of this work. Shear deformation effects were investigated for both the first order shear deformation theory and Reddy's higher order shear deformation theory for the three previously mentioned boundary conditions. Natural frequencies and buckling loads were found for various span-to-depth ratios using one inch thick square plates. Natural frequency results for the first order theory include rotatory inertia. Classical laminated plate values were also calculated when applicable. All values were calculated with M and N equal to 8. This number of terms gave converged values (less than a 1% change) as shown previously in the discussion of convergence characteristics.

Natural frequency results for the $[0/90]_s$ plate are shown on Tables 3.7 and 3.8. The span-to-depth ratio was varied from 2 to 100. Both the first order theory (FOT) and the higher order theory (HOT) normalized natural frequency values are plotted together for all three boundary conditions and are shown in Figures 3.9 ,3.10 , and 3.11. A classical laminated plate theory (CLPT) solution was found for the simply supported boundary condition and is given on the aforementioned Tables and Figures.

Nondimensional buckling loads for the $[0/90]_s$ plate are given in Tables 3.9 and 3.10. Corresponding values for each theory are plotted together in Figures 3.12, 3.13, and 3.14.

Nondimensional Natural Frequency $\bar{\omega}_n = \omega_n a^2 (\rho/E_2 h^3)^{1/2}$

First Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	4.4488	4.8870	4.5287
5	8.2404	10.8432	9.4120
10	10.6439	17.1602	15.1811
15	11.4297	20.7147	18.6607
20	11.7535	22.7727	20.6988
30	12.0107	24.8839	22.7505
50	12.1550	26.5543	24.3352
100	12.2054	28.1431	25.6213

Simple B.C. $\bar{\omega}_n$ classical = 12.2242

Table 3.7 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)

Nondimensional Natural Frequency $\bar{\omega}_n = \omega_n a^2 (\rho/E_2 h^3)^{1/2}$

Higher Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	4.4337	4.5168	4.3408
5	8.1857	9.8543	8.6893
10	10.6116	15.8196	13.9394
15	11.4116	19.3642	17.3466
20	11.7475	21.5259	19.4641
30	12.0016	23.8217	21.7156
50	12.1550	25.7222	23.5282
100	12.2054	27.4370	25.1169

Simple B.C. $\bar{\omega}_n$ classical = 12.2242

Table 3.8 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)

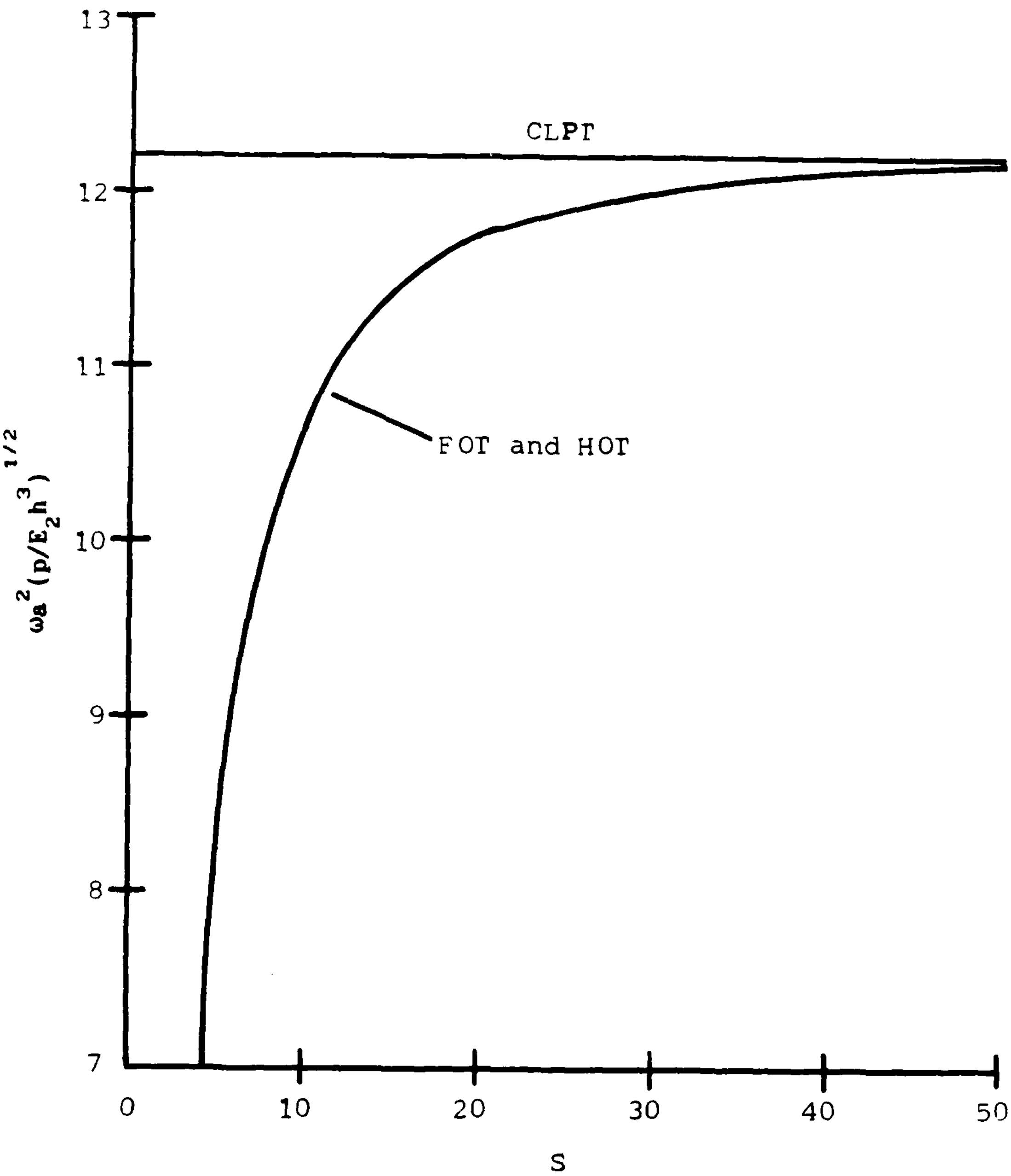

Figure 3.9 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)
Simply Supported Boundary Condition

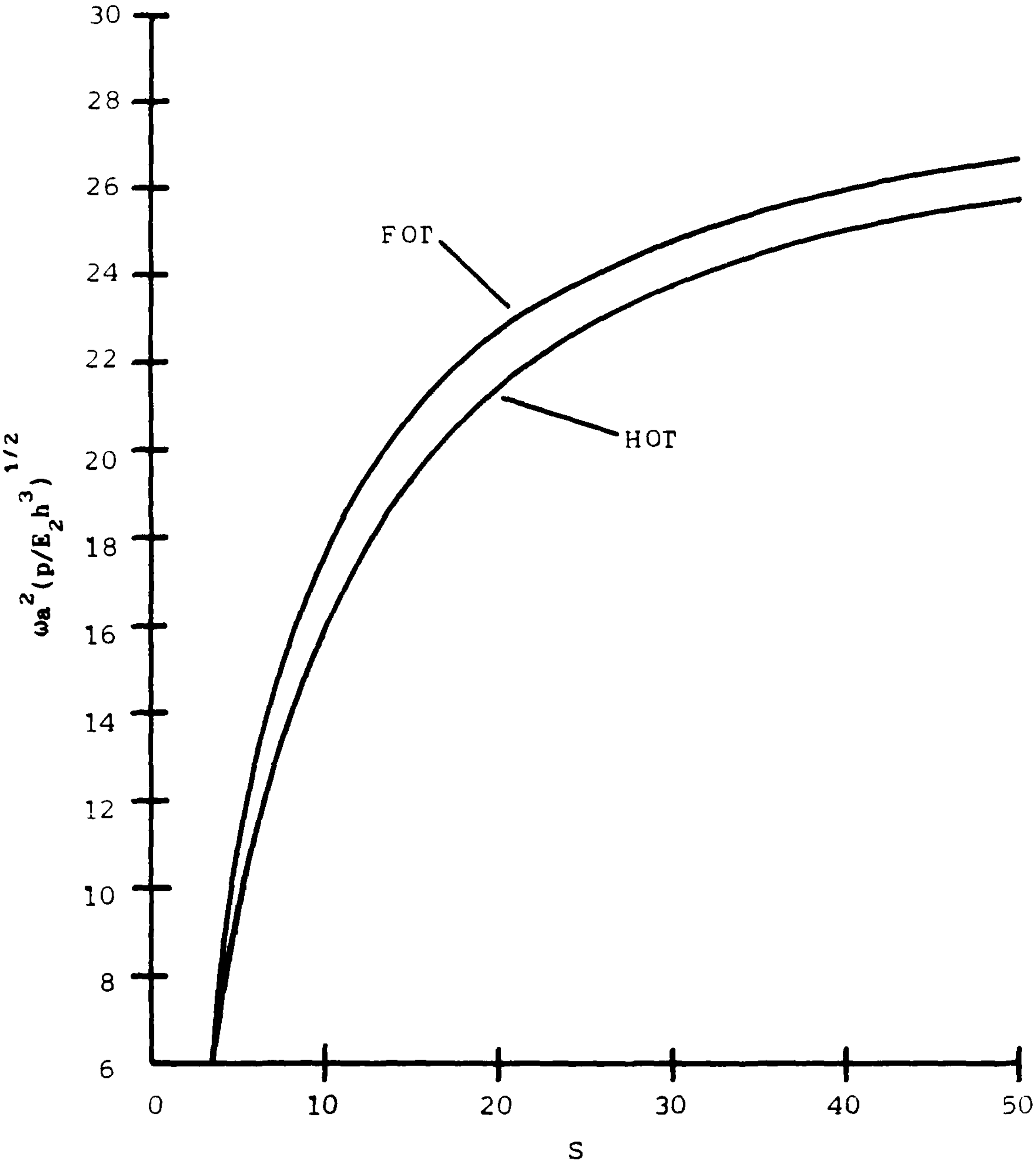

Figure 3.10 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)

Clamped Boundary Condition

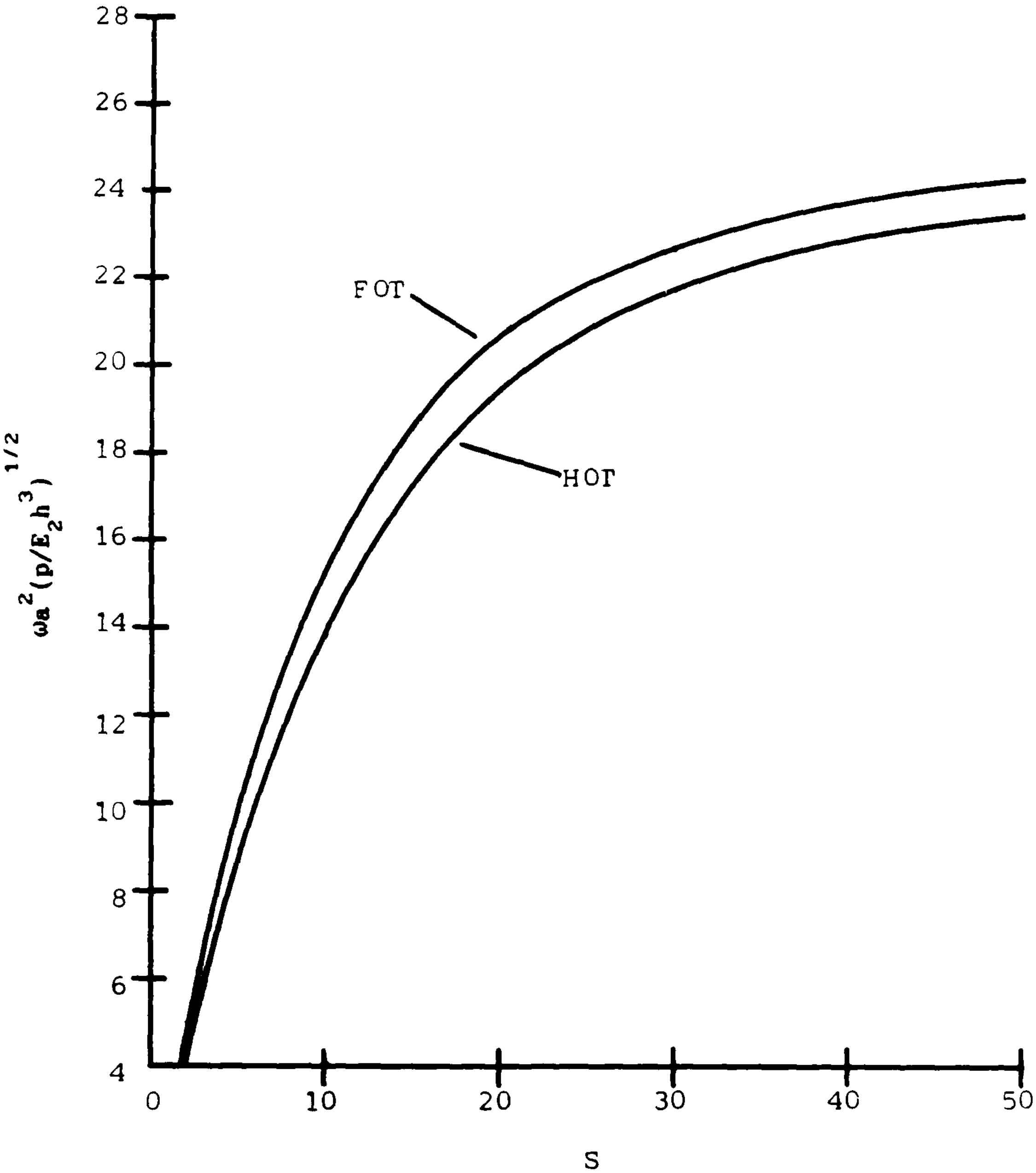

Figure 3.11 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)

Clamped-Simply Supported Boundary Condition

Nondimensional Buckling Load $\bar{N}_x = N_x a^2 / E_2 h^3$

First Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	1.3026	1.3038	1.3035
5	7.0394	8.0537	7.7585
10	11.6076	24.2766	20.8913
15	13.3157	35.7075	30.1751
20	14.0531	42.4440	35.9340
30	14.6359	49.2750	41.8288
50	14.9571	54.3571	46.1054
100	15.1000	58.5286	49.2357

Simple B.C. $\bar{N}_x$ classical = 15.1451

Table 3.9 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)

Nondimensional Buckling Load $\bar{N}_x = N_x a^2/E_2 h^3$

Higher Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	1.8286	1.8852	1.8680
5	6.9453	8.0567	7.6320
10	11.5361	22.7502	19.7886
15	13.2726	35.0645	28.2947
20	14.0260	41.4557	34.6640
30	14.6244	48.1185	40.5919
50	14.9536	53.2143	44.9875
100	15.1000	57.3286	48.2714

Simple B.C. $\bar{N}_x$ classical = 15.1451

Table 3.10 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)

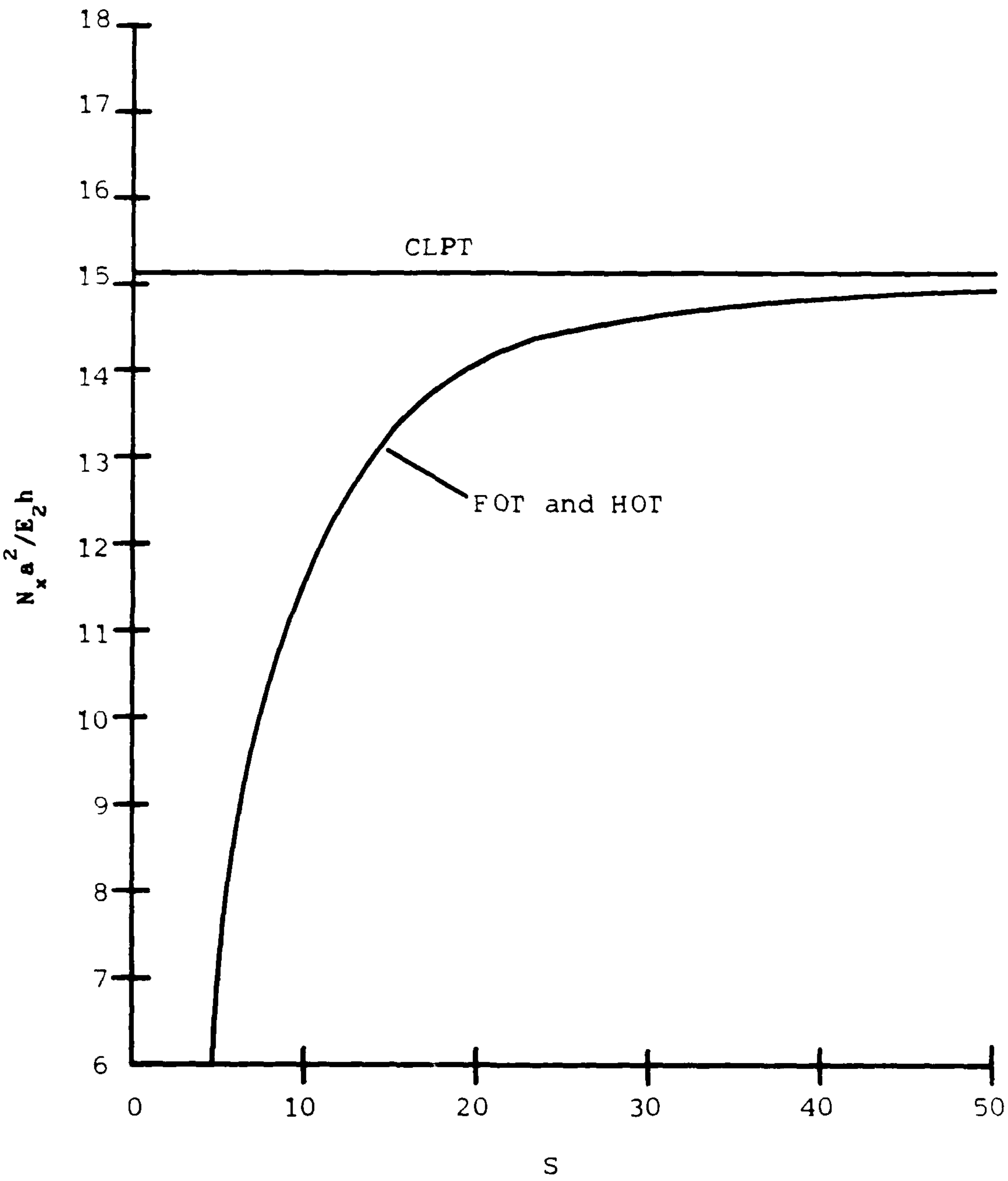

Figure 3.12 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)
Simply Supported Boundary Condition

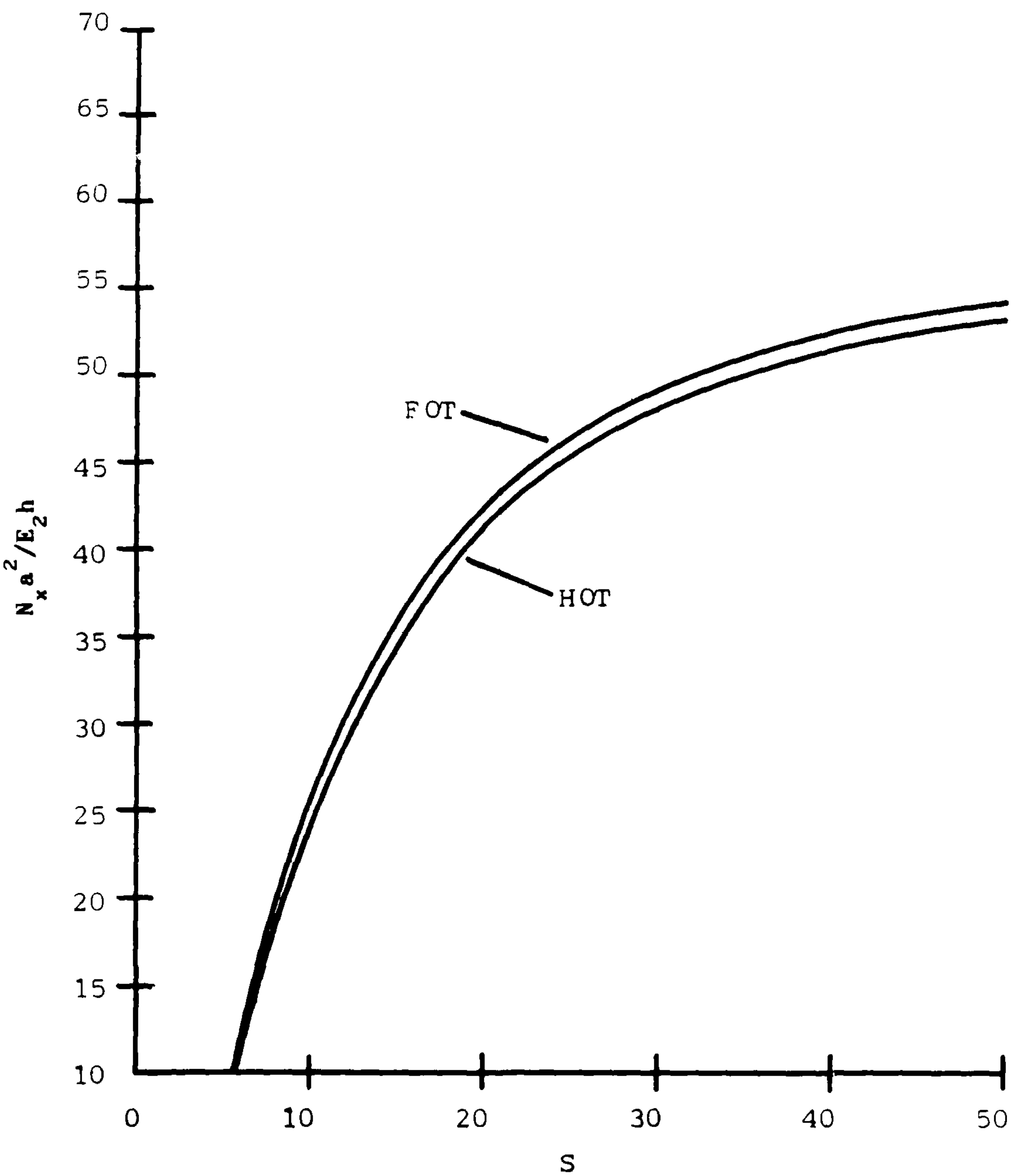

Figure 3.13 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)
Clamped Boundary Condition

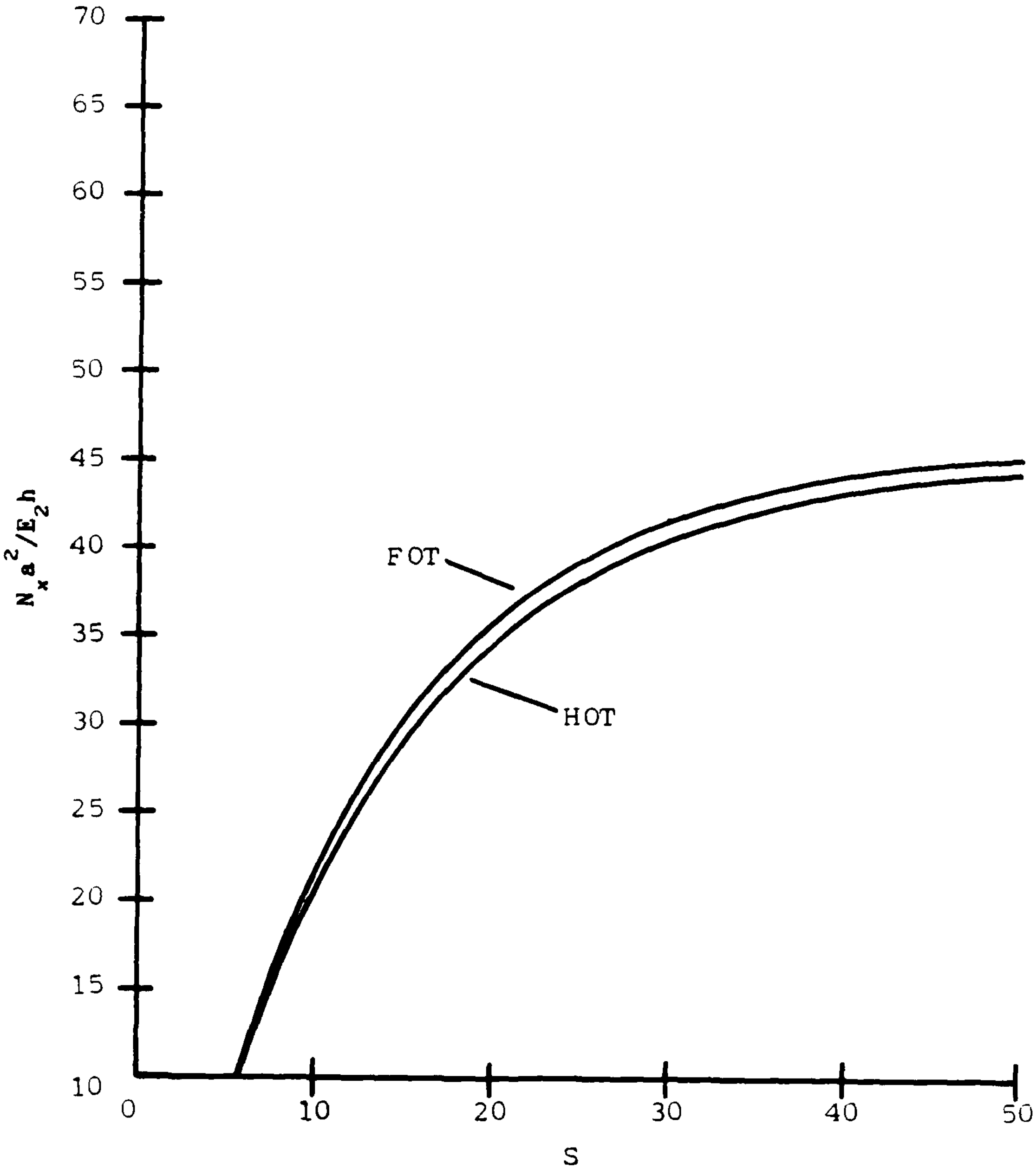

Figure 3.14 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[0/90]_s$ Plate (M=N=8)
Clamped-Simply Supported Boundary Condition

Natural frequency results for the $[\pm 45]_s$ plate are found in Tables 3.11 and 3.12. No classical laminated plate theory solutions were availible for this laminate. These values are plotted on Figures 3.15, 3.16, and 3.17.

Buckling loads can be found in Tables 3.13 and 3.14 while the plots are shown in Figures 3.18, 3.19, and 3.20 for the $[\pm 45]_s$ plate.

Nondimensional Natural Frequency $\bar{\omega}_n = \omega_n a^2 (p/E_2 h^3)^{1/2}$

First Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	4.7591	4.5670	4.7688
5	9.5424	10.6038	10.0405
10	12.7199	16.3724	14.7968
15	13.7719	19.4028	16.9925
20	14.2107	21.1183	18.1366
30	14.5618	22.8957	19.2553
50	14.7524	24.4360	20.1742
100	14.8281	26.1256	21.1829

Table 3.11 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)

Nondimensional Natural Frequency $\bar{\omega}_n = \omega_n a^2 (p/E_2 h^3)^{1/2}$

Higher Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	4.9270	4.6717	4.7867
5	9.5147	9.9661	9.6317
10	12.6321	15.5402	14.0302
15	13.6698	18.5109	16.3026
20	14.1058	20.2025	17.4588
30	14.4437	21.9879	18.6198
50	14.6263	23.5282	19.5942
100	14.7272	25.2178	20.5777

Table 3.12 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[\pm45]_s$ Plate (M=N=8)

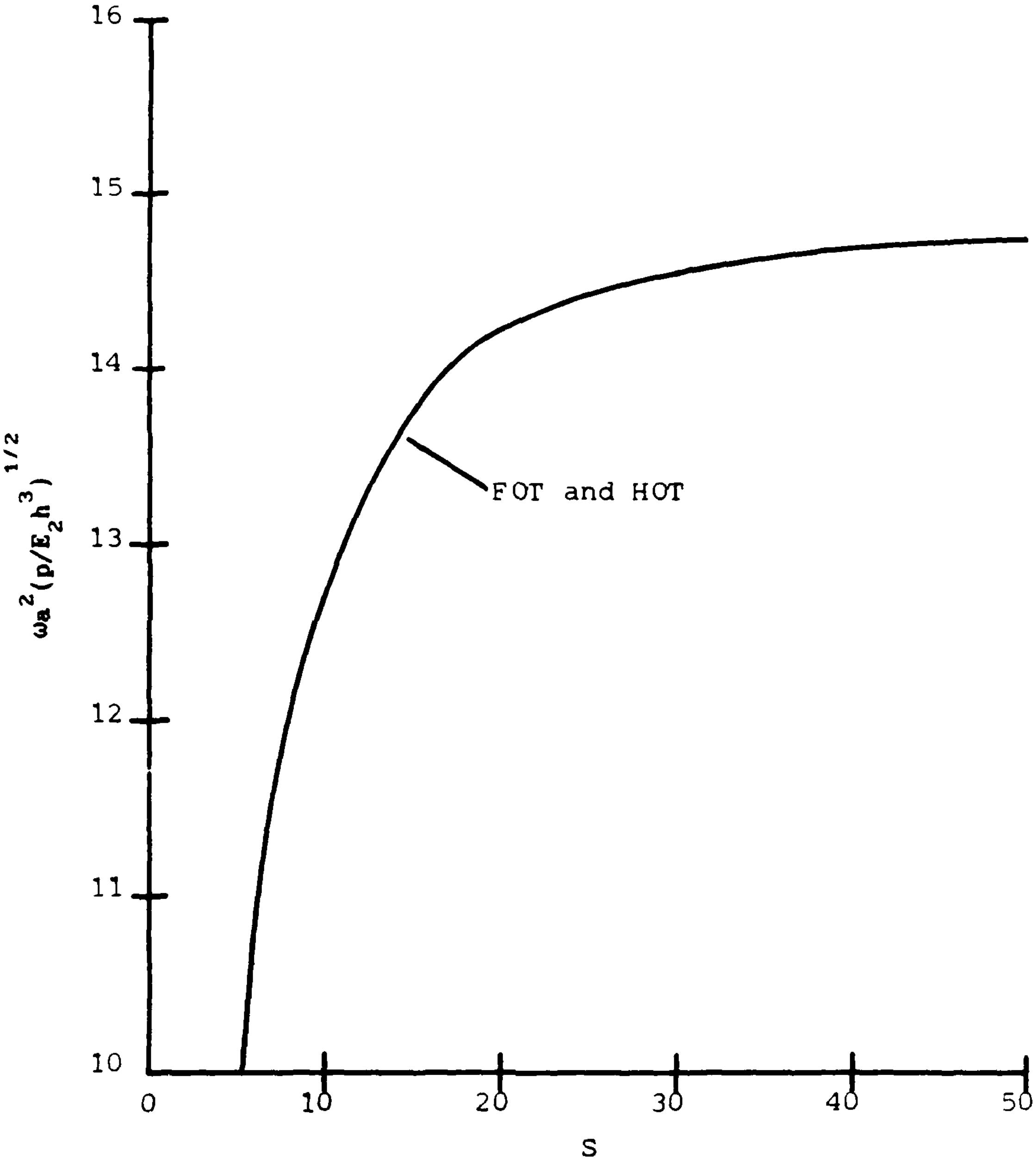

Figure 3.15 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)
Simply Supported Boundary Condition

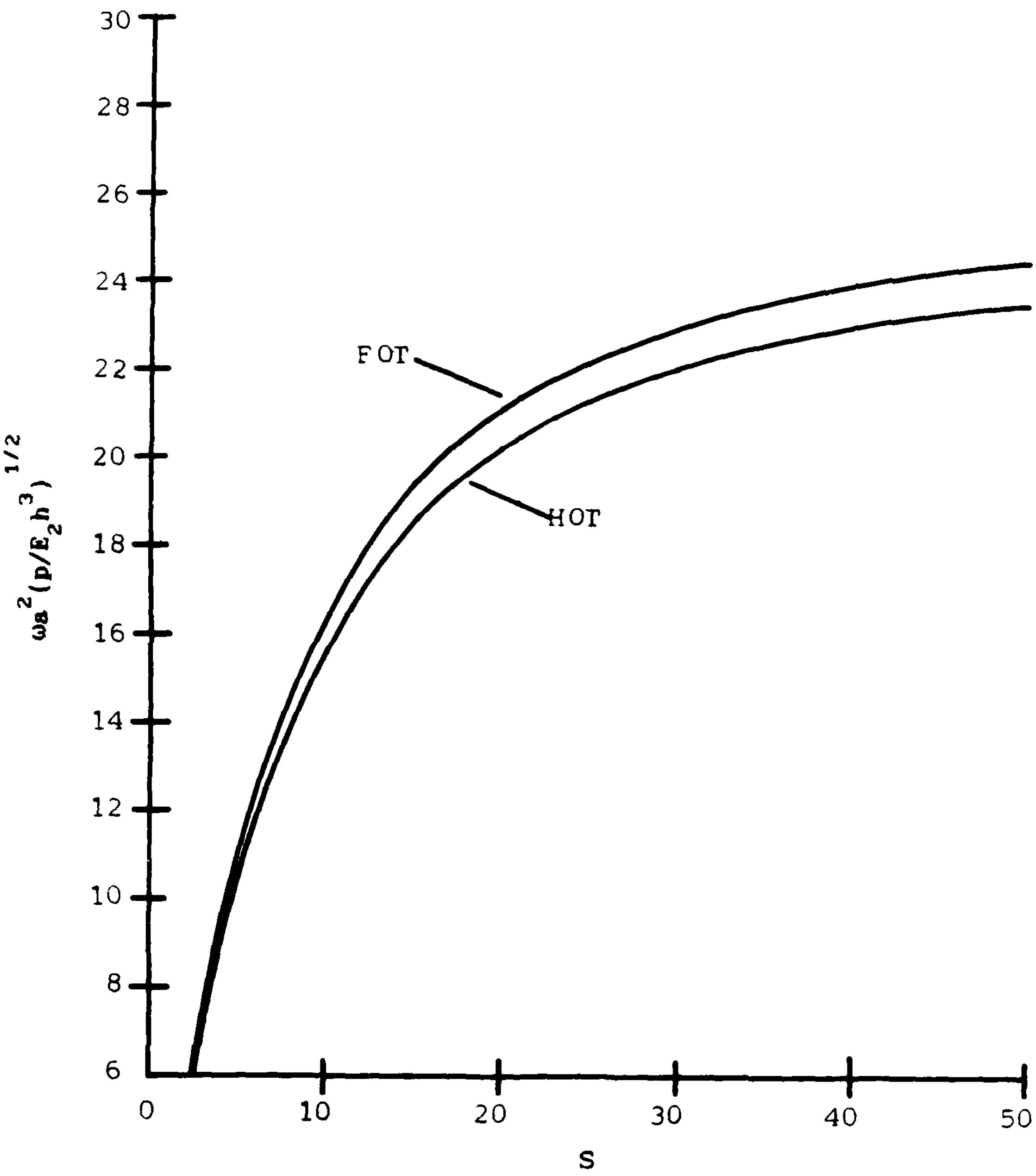

Figure 3.16 Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)

Clamped Boundary Condition

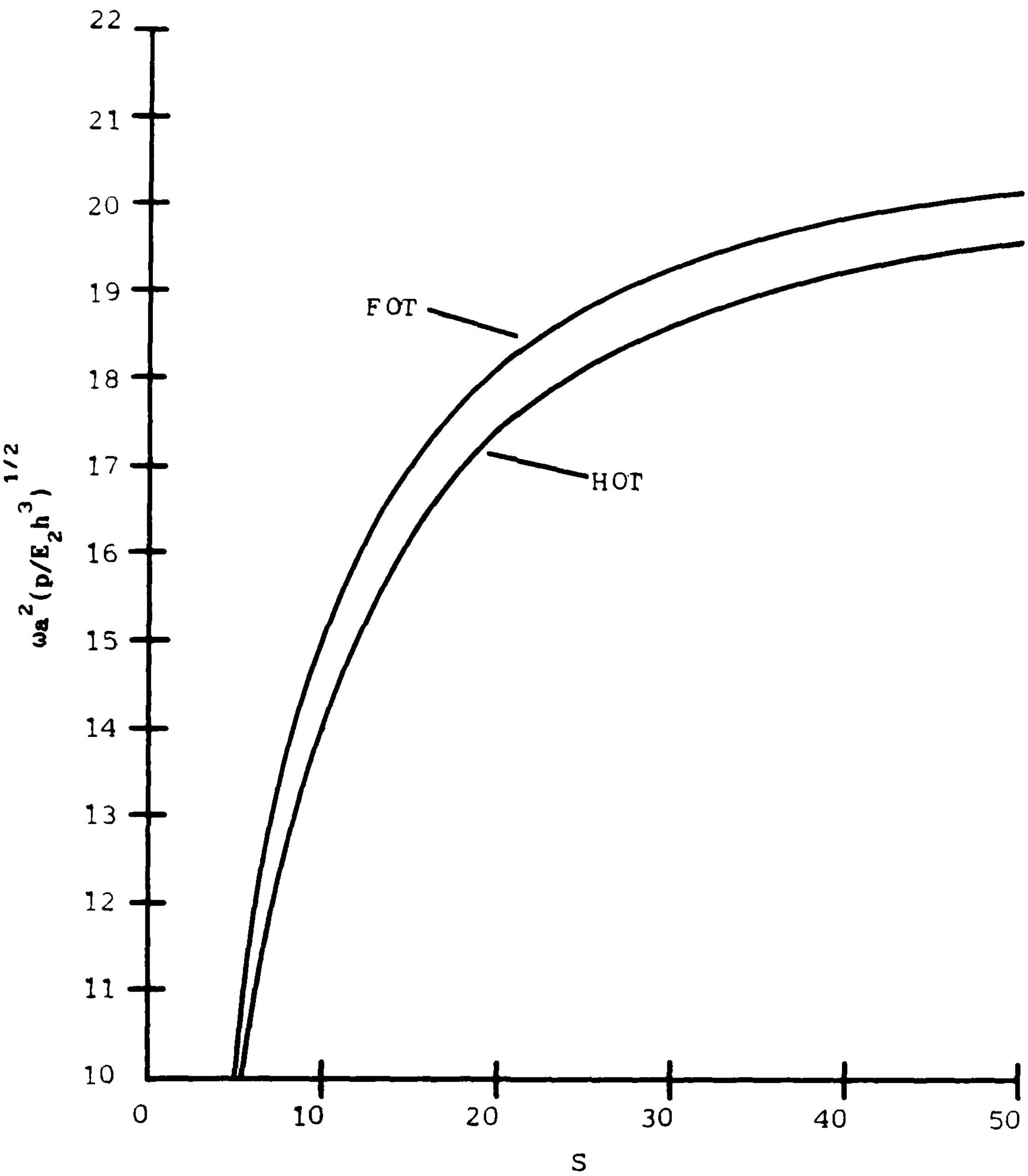

Figure 3.17 **Nondimensional Natural Frequencies vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)**

Clamped-Simply Supported Boundary Condition

Nondimensional Buckling Load $\bar{N}_x = N_x a^2 / E_2 h^3$

First Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	1.2987	1.2985	1.3012
5	7.2481	7.1291	7.4365
10	15.9054	18.2249	17.5164
15	18.8298	22.0244	22.4229
20	19.6969	30.7034	24.8960
30	20.6184	35.5339	27.1704
50	21.1339	39.4786	28.7875
100	21.3000	43.3714	30.1500

Table 3.13 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)

Nondimensional Buckling Load $\bar{N}_x = N_x a^2 / E_2 h^3$

Higher Order Theory

s	Simple B.C.	Clamped B.C.	Clamped-Simple B.C.
2	1.1793	1.5974	1.6866
5	7.0885	6.8358	7.1979
10	14.6149	17.1352	16.5975
15	18.1418	24.3731	21.2538
20	19.3114	28.6386	23.6520
30	20.2320	33.1206	25.8840
50	20.7464	36.8036	27.4768
100	20.9714	40.4286	28.8000

Table 3.14 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)

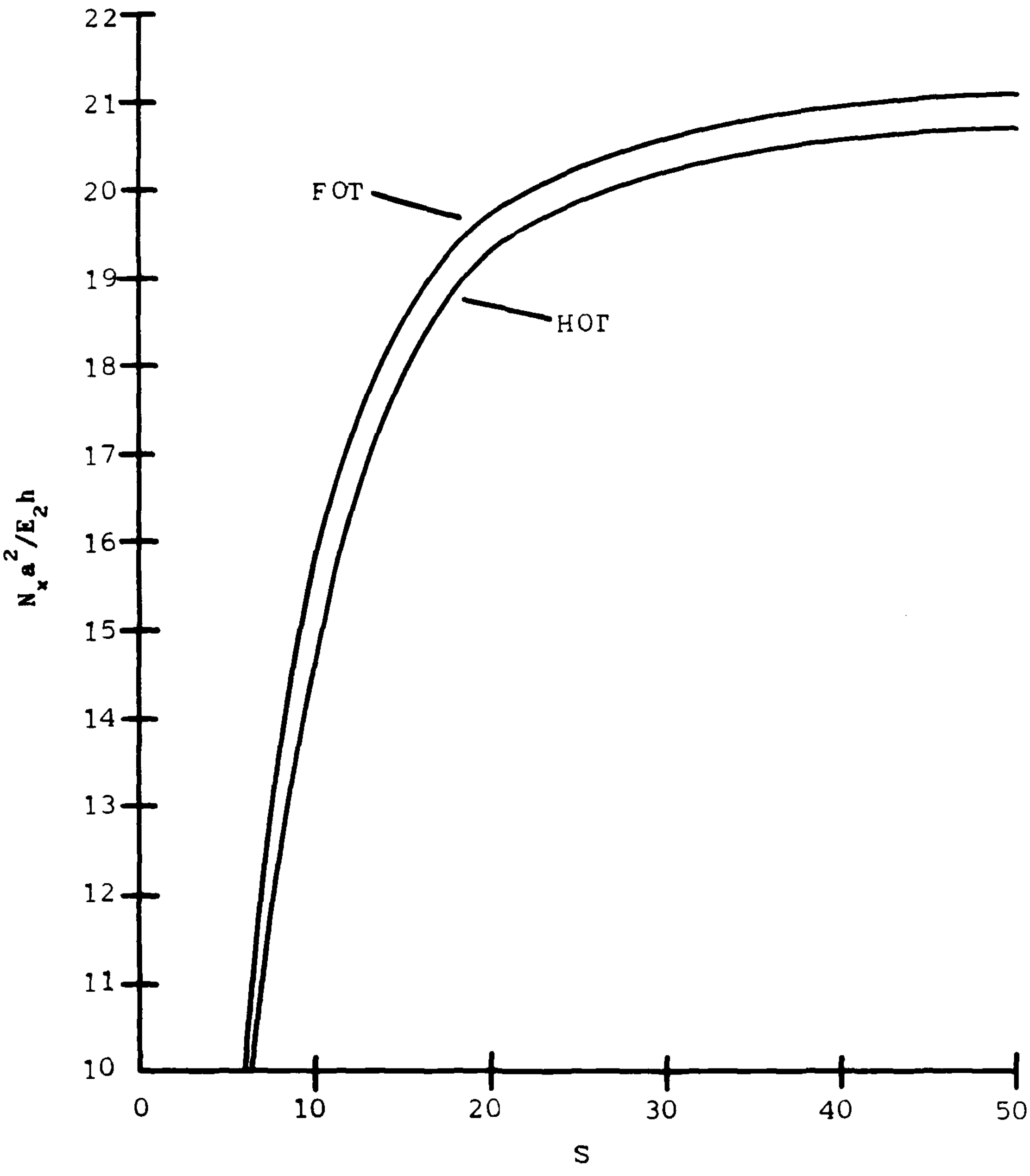

Figure 3.18 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)
Simply Supported Boundary Condition

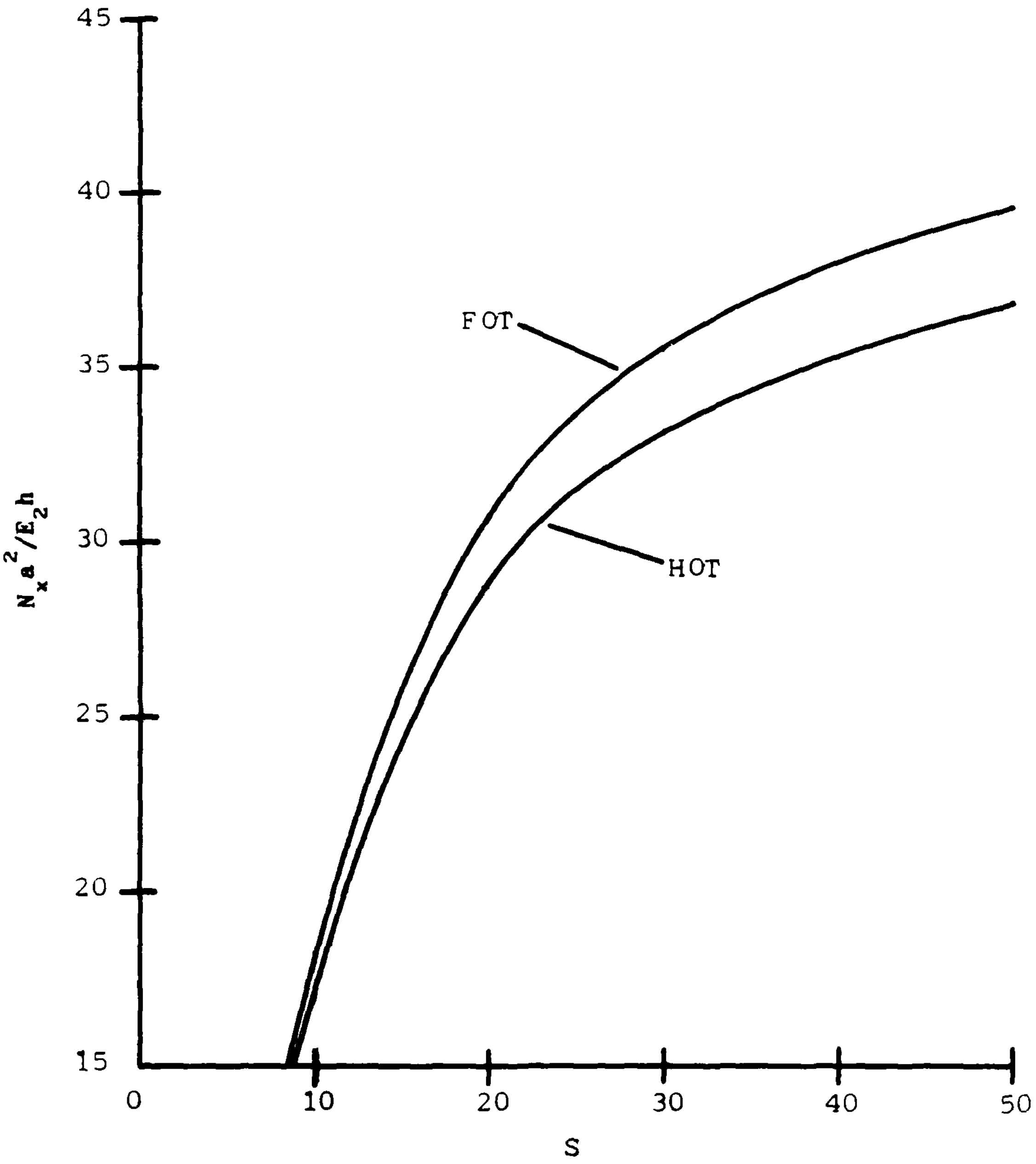

Figure 3.19 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)
Clamped Boundary Condition

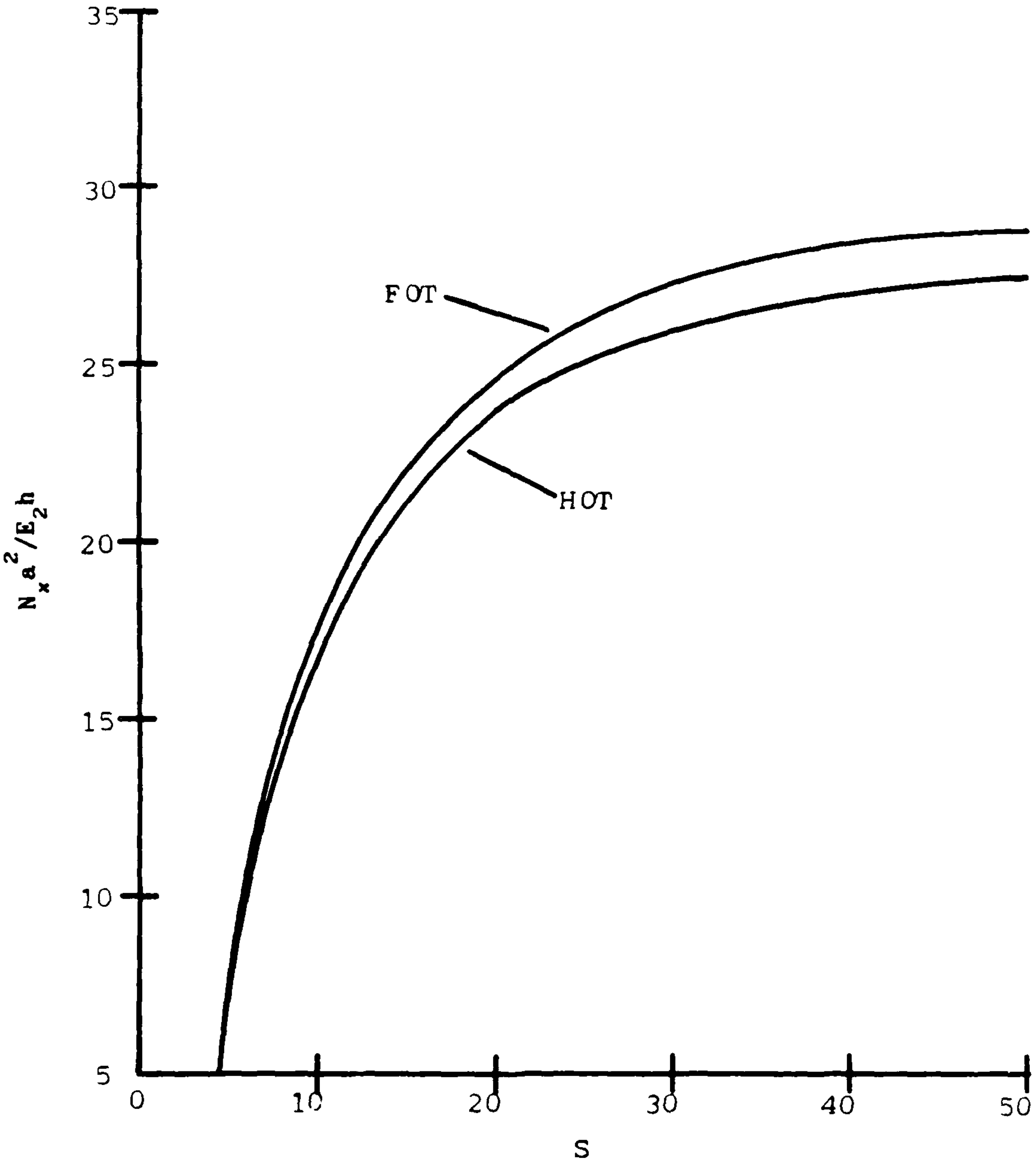

Figure 3.20 Nondimensional Buckling Loads vs Span-to-Depth Ratio for the $[\pm 45]_s$ Plate (M=N=8)

Clmaped-Simply Supported Boundary Condition

Some general observations may be made on the above data. For the natural frequencies of the $[0/90]_s$ plate with simply supported boundaries, there is very little difference between the first order theory and the high order theory (less than 1%). The first order results are seen to agree quite well with [20]. However, both theories differ greatly from classical laminated plate theory as the plate becomes thicker. Differences of 64 percent at s=2, 33 percent at s=5, 13 percent at s=10, and 6 percent at s=15 are observed. The shear deformation theory values quickly approach the classical value as the plate becomes thinner. There is less than a 1 percent difference for both theories at s=100.

The clamped boundaries of the $[0/90]_s$ plate in vibration reveal differences between the two deformation theories. The high order theory is consistently lower in value than the first order theory indicating a higher compliance. Differences of 7 to 9 percent occur for s less than 20, and 3 to 5 percent for s greater than 20. Similar variations are noticed for the clamped-simply supported boundaries. An anomaly occurs in the higher order values at s=2 wherein the simply supported natural frequency is higher than the clamped-simple natural frequency. This is obviously incorrect because the clamped-simple boundary has a greater constraint on the motion of the plate and should raise the natural frequency over the simply supported value. This occurance is probably an example of the assumption of plane stress being invalid and the two dimensional theory breaking down. The clamped and clamped-simple boundaries affect the natural frequencies to a large extent. At s=100, the clamped values differ from the simple values by 56% while the

clamped-simple values differ by 51%.

Nondimensional buckling loads for the $[0/90]_s$ plate using the higher order theory are seen to be very close to the first order loads with simple supports for s greater than 2. At s=2, the higher order values are greater by 29%. Differences from the classical plate value are 88% at s=2, 54% at s=5, 24% at s=10, 12% at s=15, and approaches very close at s=100. Higher order results run about 2% lower than first order results with clamped boundaries and about 3% lower with clamped-simple boundaries. Large differences are evident at s=2 with the higher order results being greater by 31% and 30% for the clamped and clamped-simple cases respectively. Clamped and clamped-simple frequencies exceed those of the simple boundary by 74% and 69% at s=100.

Moving on to the $[\pm45]_s$ plate, we see that again, the first order results compare well with the higher order results for simply supported natural frequencies. No classical laminated plate natural frequency is availible for this laminate. The first order theory results also agree well with [20]. Differences of around 4% between the two theories can be seen for the clamped and clamped-simple boundaries with the higher order theory being lower in value except at s=2. Frequencies for both theories are questionable at s=2 because the increased restraint does not always give higher values. A difference of 42% for the clamped boundary and a difference of 28% for the clamped-simple boundary is noted from the simply supported values at s=100. These percentages are lower than what was seen for the $[0/90]_s$ natural frequencies.

Higher order buckling loads for the $[\pm45]_s$ plate can be seen to deviate by only 3% lower than the first order loads with a simply

supported boundary. As for the natural frequencies, no classical laminated plate buckling load is availible. At s=2 the higher order value is lower by 9%. The clamped boundary loads using the higher order theory are lower on average by 6% from the first order results. The higher order loads are greater by 19% at s=2. Higher order theory loads are lower than the first order loads by 5% at s=5 or greater and higher by 23% at s=2. Values for both theories are dubious at s=2 and s=5 because of the contradiction with constraint. Loads with clamped and clamped-simple boundaries differ from the simple loads by 48% and 27% respectively. These values are again lower than the [0/90] values as also noticed for natural frequencies.

IV. Conclusions

To conclude this work, we will summarize the discussion of the convergence characteristics of the Galerkin method. The discussions of the higher order shear deformation effects will then be organized into comments on natural frequencies and comments on buckling loads. Separate comments regarding each plate will be discussed last.

The Galerkin technique was found to be a viable method for solution of the equations of motion for plates which include shear deformation. Excellent convergence was obtained for the first order theory. Convergence was good for the higher order theory, but could be improved by using double precision computer code. All cases had converged results for M=N=8. All but one case converged at M=N=6. Natural frequency calulations converged quicker than the buckling calculations. The $[0/90]_s$ plate natural frequencies and buckling loads converged quicker than the $[\pm 45]_s$ plate frequencies and loads.

Natural frequency solutions were found for both plates using both the higher order and first order shear deformation theories. The higher order natural frequencies are lower than the first order solutions by a maximum of 6 percent. Both theories predict similar natural frequencies for a simply supported plate. The higher order theory is within 10 percent of the classical laminated plate natural frequency for span-to-depth ratios greater than 15. A difference of 64 percent can occur at s=2.

Both theories were able to calculate buckling loads for the $[0/90]_s$ and $[\pm 45]_s$ plates. Buckling loads using the higher order theory

are lower than the first order theory loads by a maximum of 6 percent for s greater than 2. At s=2, the higher order buckling loads are greater than the first order loads by 30 percent for the $[0/90]_s$ plate and by 15 percent for the $[\pm45]_s$ plate. The classical laminated plate buckling load is within 10 percent of the higher order theory for s greater than 20. At s=2, a difference of 88 percent was seen.

Finally, the higher order theory shows that the $[0/90]_s$ plate is more boundary sensitive than the $[\pm45]_s$ plate. Greater differences between simply supported results and clamped and clamped-simple results were found for the $[0/90]_s$ plate. Also, the clamped and clamped-simple values are closer with the $[0/90]_s$ plate than with the $[\pm45]_s$ plate. Lastly, the $[\pm45]_s$ plate displays a narrower band of values for all three boundary conditions than does the $[0/90]_s$ plate.

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VITA

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[redacted]

Appendix A

```
1 REM   COMPOSITE PLATE STIFFNESS PROGRAM
5 A44=0 : A45=0 : A55=0 : D44=0 : D45=0 : D55=0 : F44=0 : F45=0 : F55=0
: P=0
10 D11=0 : D12=0 : D22=0 : D16=0 : D26=0 : D66=0
15 F11=0 : F12=0 : F22=0 : F16=0 : F26=0 : F66=0
20 H11=0 : H12=0 : H22=0 : H16=0 : H26=0 : H66=0
50 INPUT"ADD ANOTHER LAYER";N$
60 IF N$="N" THEN 1010
100 REM
110 REM INPUT LAMINATE DATA
120 INPUT"ORIENTATION ANGLE";AN
130 TH=AN*3.1415927#/180
140 REM
150 INPUT"ZK-1 DIMENSION";ZK1
160 INPUT"ZK DIMENSION";ZK
170 REM
180 INPUT"SAME MATERIAL";M$
190 IF M$="Y" THEN 370
200 INPUT"E11";E11
210 INPUT"E22";E22
220 INPUT"V12";V12
230 V21=V12*(E22/E11)
240 INPUT"G12";G12
250 G13=G12
260 G23=.8*G12
270 INPUT"DENSITY OF PLY";RHO
280 REM
290 REM COMPUTE Q'S
300 Q11=E11/(1-V12*V21)
310 Q12=(V12*E22)/(1-V12*V21)
320 Q22=E22/(1-V12*V21)
330 Q44=G23
340 Q55=G13
350 Q66=G12
360 REM
370 REM COMPUTE Q BAR'S
380 QB11=Q11*COS(TH)^4+2*(Q12+2*Q66)*SIN(TH)^2*COS(TH)^2+Q22*SIN(TH)^4
390 QB12=(Q11+Q22-4*Q66)*SIN(TH)^2*COS(TH)^2+Q12*(SIN(TH)^4+COS(TH)^4)
400 QB22= Q11*SIN(TH)^4+2*(Q12+2*Q66)*SIN(TH)^2*COS(TH)^2+Q22*COS(TH)^4
410 QB16=(Q11-Q12-2*Q66)*SIN(TH)*COS(TH)^3+(Q12-Q22+2*Q66)*SIN(TH)^3*
COS(TH)
420 QB26=(Q11-Q12-2*Q66)*SIN(TH)^3*COS(TH)+(Q12-Q22+2*Q66)*SIN(TH)*
COS(TH)^3
430 QB66=(Q11+Q22-2*Q12-2*Q66)*SIN(TH)^2*COS(TH)^2+Q66*(SIN(TH)^4+
COS(TH)^4)
440 QB44=Q44*COS(TH)^2+Q55*SIN(TH)^2
450 QB45=(Q44-Q55)*COS(TH)*SIN(TH)
460 QB55=Q55*COS(TH)^2+Q44*SIN(TH)^2
470 REM
480 REM COMPUTE A STIFFNESSES
490 Z [illegible]K-ZK1
560 A44=A44+QB44*Z1
570 A45=A45+QB45*Z1
580 A55=A55+QB55*Z1
585 REM
```

```
590 P=P+RHO*Z1
600 REM
610 REM COMPUTE D STIFFNESSES
620 Z3=(1/3)*(ZK^3-ZK1^3)
630 D11=D11+QB11*Z3
640 D12=D12+QB12*Z3
650 D22=D22+QB22*Z3
660 D16=D16+QB16*Z3
670 D26=D26+QB26*Z3
680 D66=D66+QB66*Z3
690 D44=D44+QB44*Z3
700 D45=D45+QB45*Z3
710 D55=D55+QB55*Z3
720 REM
730 REM COMPUTE F STIFFNESSES
740 Z5=(1/5)*(ZK^5-ZK1^5)
750 F11=F11+QB11*Z5
760 F12=F12+QB12*Z5
770 F22=F22+QB22*Z5
780 F16=F16+QB16*Z5
790 F26=F26+QB26*Z5
800 F66=F66+QB66*Z5
810 F44=F44+QB44*Z5
820 F45=F45+QB45*Z5
830 F55=F55+QB55*Z5
840 REM
850 REM COMPUTE H STIFFNESSES
860 Z7=(1/7)*(ZK^7-ZK1^7)
870 H11=H11+QB11*Z7
880 H12=H12+QB12*Z7
890 H22=H22+QB22*Z7
900 H16=H16+QB16*Z7
910 H26=H26+QB26*Z7
920 H66=H66+QB66*Z7
930 H44=H44+QB44*Z7
940 H45=H45+QB45*Z7
950 H55=H55+QB55*Z7
960 REM
970 GOTO 50
1000 REM
1010 REM PRINT OUT STIFFNESSES
1020 LPRINT "A44,A45,A55",A44,A45,A55
1030 LPRINT "D44,D45,D55",D44,D45,D55
1040 LPRINT "F44,F45,F55",F44,F45,F55
1070 REM
1075 LPRINT " "
1080 LPRINT "D11,D12,D16",D11,D12,D16
1090 LPRINT "    D22,D26","                    ",D22,D26
1100 LPRINT "        D66","                                        ",D66
1110 REM
1115 LPRINT " "
1120 LPRINT "F11,F12,F16",F11,F12,F16
1130 LPRINT "    F22,F26","                    ",F22,F26
1140 LPRINT "        F66","                                        ",F66
1150 REM
```

```
1155 LPRINT " "
1160 LPRINT "H11,H12,H16",H11,H12,H16
1170 LPRINT "    H22,H26","                  ",H22,H26
1180 LPRINT "        H66","                                    ",H66
1190 REM
1200 REM COMPUTE NORMALIZED STIFFNESSES
1210 INPUT"PLATE THICKNESS";H
1215 REM
1220 AN=1/(E22*H)
1290 A44=A44*AN
1300 A45=A45*AN
1310 A55=A55*AN
1315 REM
1320 DN=1/(E22*H^3)
1330 D11=D11*DN
1340 D12=D12*DN
1350 D22=D22*DN
1360 D16=D16*DN
1370 D26=D26*DN
1380 D66=D66*DN
1390 D44=D44*DN
1400 D45=D45*DN
1410 D55=D55*DN
1420 REM
1430 NF=1/(E22*H^5)
1440 F11=F11*NF
1450 F12=F12*NF
1460 F22=F22*NF
1470 F16=F16*NF
1480 F26=F26*NF
1490 F66=F66*NF
1500 F44=F44*NF
1510 F45=F45*NF
1520 F55=F55*NF
1530 REM
1540 HN=1/(E22*H^7)
1550 H11=H11*HN
1560 H12=H12*HN
1570 H22=H22*HN
1580 H16=H16*HN
1590 H26=H26*HN
1600 H66=H66*HN
1610 H44=H44*HN
1620 H45=H45*HN
1630 H55=H55*HN
1640 REM
1650 PRINT NORMALIZED STIFFNESSES
1655 LPRINT " "
1660 LPRINT "a44,a45,a55",A44,A45,A55
1670 LPRINT "d44,d45,d55",D44,D45,D55
1680 LPRINT "f44,f45,f55",F44,F45,F55
1690 REM
1695 LPRINT " "
1700 LPRINT "d11,d12,d16",D11,D12,D16
1710 LPRINT "    d22,d26","                ",D22,D26
```

```
1720 LPRINT "         d66","                                    ",D66
1730 REM
1735 LPRINT " "
1740 LPRINT "f11,f12,f16",F11,F12,F16
1750 LPRINT "    f22,f26","                ",F22,F26
1760 LPRINT "         f66","                                    ",F66
1770 REM
1775 LPRINT " "
1780 LPRINT "h11,h12,h16",H11,H12,H16
1790 LPRINT "    h22,h26","                ",H22,H26
1800 LPRINT "         h66","                                    ",H66
1805 LPRINT " "
1850 LPRINT "P",P
2000 END
```

Appendix B

```
      PROGRAM GALERK1
C     SIMPLY SUPPORTED BOUNDARIES
C
      DIMENSION ST(300,300)
      REAL K,K1,K2,MA(300,300),KK1,KK2,KK3,LAM1,LAM2
      INTEGER M,N,P,Q,EOTEST1,EOTEST2,MMAX,NMAX,HOT,VIB
      OPEN(5,FILE='GALIN')
      OPEN(6,FILE='EIGIN')
C
C     DEFINE CONSTANTS
      K=5.0/6.0
      PI=3.1415926535
      PI2=PI**2
      PI3=PI**3
      PI4=PI**4
C
C     READ INPUT DATA
C     HIGHER ORDER THEORY, VIBRATION, AND ROTATORY INERTIA FLAGS
      READ(5,*) HOT,VIB,ROT
      READ(5,*) MMAX,NMAX
      READ(5,*) A,B,H
      READ(5,*) RHO,ET,K1,K2,KK1,KK2,KK3
      READ(5,*) A44,A45,A55
      READ(5,*) D44,D45,D55
      READ(5,*) F44,F45,F55
      READ(5,*) D11,D12,D16,D22,D26,D66
      READ(5,*) F11,F12,F16,F22,F26,F66
      READ(5,*) H11,H12,H16,H22,H26,H66
C
C     INTRODUCE SHEAR CORRECTION TERMS (FOR FIRST ORDER THEORIES)
      IF(HOT.EQ.1) GOTO 5
      IF(K1.EQ.0.0) THEN
C       SINGLE COEFFICIENT
        A44=K*A44
        A55=K*A55
      ELSE
C       DOUBLE COEFFICIENT
        A44=K2*A44
        A55=K1*A55
      ENDIF
C
C     CALCULATE ASPECT RATIO, SPAN-TO-DEPTH RATIO, AND MASS DENSITY
5     R=A/B
      S=A/H
      P1=RHO*H
      P1=P1/(32.1740*12.0)
C
C     FORM NESTED LOOPS TO GENERATE GALERKIN TERMS
      I=1
      J=1
      DO 10 P=1,MMAX
      DO 10 Q=1,NMAX
      DO 20 M=1,MMAX
      DO 20 N=1,NMAX
C
```

```
C     PERFORM INTEGRATION
C
      EOTEST1=MOD(M+P,2)
      EOTEST2=MOD(N+Q,2)
      IF(M.EQ.P) THEN
        F1=0.5
        F3=0.0
        F5=0.0
      ELSE
        F1=0.0
        IF(EOTEST1.EQ.0) THEN
          F3=0.0
          F5=0.0
        ELSE
          F3=2.0*P/(PI*(P**2-M**2))
          F5=2.0*M/(PI*(M**2-P**2))
        ENDIF
      ENDIF
      IF(N.EQ.Q) THEN
        F2=0.5
        F4=0.0
        F6=0.0
      ELSE
        F2=0.0
        IF(EOTEST2.EQ.0) THEN
          F4=0.0
          F6=0.0
        ELSE
          F4=2.0*Q/(PI*(Q**2-N**2))
          F6=2.0*N/(PI*(N**2-Q**2))
        ENDIF
      ENDIF
C
C     COMPUTE STIFFNESS TERMS
C
      A1=F1*F2*(S**2*M**2*PI2)*(-D11+8./3.*F11-16./9.*H11)+F1*F2*(S**2*R
     +**2*N**2*PI2)*(-D66+8./3.*F66-16./9.*H66)+F1*F2*(S**4)*(-A55+8.*D5
     +5-16.*F55)+F4*F5*(S**2*R*M*N*PI2)*(-2.*D16+16./3.*F16-32./9.*H16)+
     +F4*(1.-COS(M*PI)*COS(P*PI))*(S**2*R*N*PI)*(D16-4.*F16+32./9.*H16)
C
      B1=F1*F2*(S**2*R*M*N*PI2)*(-D12+8./3.*F12-D66+8./3.*F66-16./9.*H12
     +-16./9.*H66)+F5*F4*(S**2*M**2*PI2)*(-D16+8./3.*F16-16./9.*H16)+F5*
     +F4*(S**4)*(-A45+8.*D45-16.*F45)+F5*F4*(S**2*R**2*N**2*PI2)*(-D26+8
     +./3.*F26-16./9.*H26)+F4*(1.-COS(M*PI)*COS(P*PI))*(S**2*M*PI)*(D16-
     +4.*F16+32./9.*H16)
C
      C1=F1*F2*(S*R**2*M*N**2*PI3)*(4./3.*F12+8./3.*F66-16./9.*H12-32./9
     +.*H66)+F1*F2*(S*M**3*PI3)*(4./3.*F11-16./9.*H11)+F1*F2*(S**3*M*PI)
     +*(-A55+8.*D55-16.*F55)+F5*F4*(S*R*M**2*N*PI3)*(4.*F16-48./9.*H16)+
     +F5*F4*(S*R**3*N**3*PI3)*(4./3.*F26-16./9.*H26)+F5*F4*(S**3*R*N*PI)
     +*(-A45+8.*D45-16.*F45)+F4*(1.-COS(M*PI)*COS(P*PI))*(S*R*M*N*PI2)*(
     +-8./3.*F16+64./9.*H16)
C
      A2=F1*F2*(S**2*R*M*N*PI2)*(-D12+8./3.*F12-D66+4./3.*F26+8./3.*F66-
     +16./9.*H12-16./9.*H66)+F3*F6*(S**2*R**2*N**2*PI2)*(-D26+8./3.*F26
```

```
     +-16./9.*H26)+F3*F6*(S**2*M**2*PI2)*(-D16+8./3.*F16-16./9.*H16)+F3*
     +F6*(S**4)*(-A45+8.*D45-16.*F45)+F3*(1.-COS(N*PI)*COS(Q*PI))*(S**2*
     +R*N*PI)*(D26-4.*F26+32./9.*H26)
C
      B2=F1*F2*(S**2*R**2*N**2*PI2)*(-D22+8./3.*F22-16./9.*H22)+F1*F2*(S
     +**2*M**2*PI2)*(-D66+8./3.*F66-16./9.*H66)+F1*F2*(S**4)*(-A44+8.*D4
     +4-16.*F44)+F3*F6*(S**2*R*M*N*PI2)*(-2.*D26+4.*F26-32./9.*H26)+F3*(
     +1.-COS(N*PI)*COS(Q*PI))*(S**2*M*PI)*(D26-4.*F26+32./9.*H26)
C
      C2=F1*F2*(S*R*M**2*N*PI3)*(4./3.*F12+8./3.*F66-16./9.*H12-32./9.*H
     +66)+F1*F2*(S*R**3*N**3*PI3)*(4./3.*F22-16./9.*H22)+F1*F2*(S**3*R*N
     +*PI)*(-A44+8.*D44-16.*F44)+F3*F6*(S*R**2*M*N**2*PI3)*(4.*F26-48./9
     +.*H26)+F3*F6*(S*M**3*PI3)*(4./3.*F16-16./9.*H16)+F3*F6*(S**3*M*PI)
     +*(-A45+8.*D45-16.*F45)+F3*(1.-COS(N*PI)*COS(Q*PI))*(S*R*M*N*PI2)*(
     +-8./3.*F26+64./9.*H26)
C
      A3=F1*F2*(S**3*M*PI)*(-A55+8.*D55-16.*F55)+F1*F2*(S*M**3*PI3)*(4./
     +3.*F11-16./9.*H11)+F1*F2*(S*R**2*M*N**2*PI3)*(4./3.*F12-16./9.*H12
     ++8./3.*F66-32./9.*H66)+F3*F4*(S**3*R*N*PI)*(A45-8.*D45+16.*F45)+F3
     +*F4*(S*R*M**2*N*PI3)*(-4.*F16+48./9.*H16)+F3*F4*(S*R**3*N**3*PI3)*
     +(-4./3.*F26+16./9.*H26)
C
      B3=F1*F2*(S**3*R*N*PI)*(-A44+8.*D44-16.*F44)+F1*F2*(S*R*M**2*N*PI3
     +)*(4./3.*F12-16./9.*H12+8./3.*F66-32./9.*H66)+F1*F2*(S*R**3*N**3*P
     +I3)*(4./3.*F22-16./9.*H22)+F3*F4*(S**3*M*PI)*(A45-8.*D45+16.*F45)+
     +F3*F4*(S*M**3*PI3)*(-4./3.*F16+16./9.*H16)+F3*F4*(S*R**2*M*N**2*PI
     +3)*(-4.*F26+48./9.*H26)
C
      C3=F1*F2*(S**2*M**2*PI2)*(-A55+8.*D55-16.*F55)+F1*F2*(S**2*R**2*N*
     +*2*PI2)*(-A44+8.*D44-16.*F44)+F1*F2*(PI4)*(-16./9.*H11*M**4-32./9.
     +*H12*R**2*M**2*N**2-16./9.*H22*R**4*N**4-64./9.*H66*R**2*M**2*N**2
     +)+F3*F4*(2.*S**2*R*M*N*PI2)*(A45-8.*D45+16.*F45)+F3*F4*(64./9.*PI4
     +)*(H16*R*M**3*N+H26*R**3*M*N**3)
C
C     STORE STIFFNESS TERMS IN ST MATRIX
C
      ST(I,J)=-A1
      ST(I,J+MMAX*NMAX)=-B1
      ST(I,J+2*MMAX*NMAX)=-C1
      ST(I+MMAX*NMAX,J)=-A2
      ST(I+MMAX*NMAX,J+MMAX*NMAX)=-B2
      ST(I+MMAX*NMAX,J+2*MMAX*NMAX)=-C2
      ST(I+2*MMAX*NMAX,J)=-A3
      ST(I+2*MMAX*NMAX,J+MMAX*NMAX)=-B3
      ST(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=-C3
C
C     COMPUTE MASS/INERTIA TERMS
C
      IF(VIB.EQ.1) THEN
C       VIBRATION MASS/INERTIA TERMS
        IF(HOT.EQ.1) THEN
C         HIGHER ORDER THEORY
          LAM1=A**2*PI/(ET*H**3)
          X1=F1*F2*(17./315.)*(S**2)*LAM1
          Y1=0.0
```

```
            Z1=F1*F2*(-4./315.)*(S*M*PI)*LAM1
            X2=0.0
            Y2=F1*F2*(17./315.)*(S**2)*LAM1
            Z2=F1*F2*(-4./315.)*(S*R*N*PI)*LAM1
            X3=F1*F2*(-4./315.)*(S*M*PI)*LAM1
            Y3=F1*F2*(-4./315.)*(S*R*N*PI)*LAM1
            Z3=F1*F2*(S**2+16./4032.*(M**2*PI2)+16./4032.*(R**2*N**2*PI2))
     +*LAM1
          ELSE
C           FIRST ORDER THEORY
            X1=ROT*F1*F2*(A**4*P1/(12.*ET*H**3))
            Y1=0.0
            Z1=0.0
            X2=0.0
            Y2=ROT*F1*F2*(A**4*P1/(12.*ET*H**3))
            Z2=0.0
            X3=0.0
            Y3=0.0
            Z3=F1*F2*(A**4*P1/(ET*H**4))
          ENDIF
        ELSE
C         BUCKLING MASS/INERTIA TERMS (IDENTICAL FOR ALL THEORIES)
          LAM2=A**2/(ET*H**3)
          X1=0.0
          Y1=0.0
          Z1=0.0
          X2=0.0
          Y2=0.0
          Z2=0.0
          X3=0.0
          Y3=0.0
          Z3=F1*F2*(KK1*M**2*PI2+R**2*KK2*N**2*PI2)*LAM2+F3*F4*(2.*R*KK3*
     +M*N*PI2)*LAM2
        ENDIF
C
C       STORE MASS/INERTIA TERMS IN MA MATRIX
C
          MA(I,J)=X1
          MA(I,J+MMAX*NMAX)=Y1
          MA(I,J+2*MMAX*NMAX)=Z1
          MA(I+MMAX*NMAX,J)=X2
          MA(I+MMAX*NMAX,J+MMAX*NMAX)=Y2
          MA(I+MMAX*NMAX,J+2*MMAX*NMAX)=Z2
          MA(I+2*MMAX*NMAX,J)=X3
          MA(I+2*MMAX*NMAX,J+MMAX*NMAX)=Y3
          MA(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=Z3
C
        J=J+1
20      CONTINUE
        I=I+1
        J=1
10      CONTINUE
C
C       WRITE STIFFNESS AND MASS/INERTIA MATRICIES TO A FILE
        WRITE(6,15) MMAX
```

```
45    FORMAT(I2)
      DO 40 I=1,3*MMAX*NMAX
      DO 40 J=1,3*MMAX*NMAX
      WRITE(6,50) ST(I,J),MA(I,J)
50    FORMAT(2E16.8)
40    CONTINUE
      STOP
      END
```

```
      PROGRAM GALERK2
C     CLAMPED BOUNDARIES
C
      DIMENSION ST(300,300)
      REAL K,K1,K2,MA(300,300),KK1,KK2,KK3,LAM1,LAM2
      INTEGER M,N,P,Q,EOTEST1,EOTEST2,MMAX,NMAX,HOT,VIB
      OPEN(5,FILE='GALIN')
      OPEN(6,FILE='EIGIN')
C
C     DEFINE CONSTANTS
      K=5.0/6.0
      PI=3.1415926535
      PI2=PI**2
      PI3=PI**3
      PI4=PI**4
C
C     READ INPUT DATA
C     HIGHER ORDER THEORY, VIBRATION, AND ROTATORY INERTIA FLAGS
      READ(5,*) HOT,VIB,ROT
      READ(5,*) MMAX,NMAX
      READ(5,*) A,B,H
      READ(5,*) RHO,ET,K1,K2,KK1,KK2,KK3
      READ(5,*) A44,A45,A55
      READ(5,*) D44,D45,D55
      READ(5,*) F44,F45,F55
      READ(5,*) D11,D12,D16,D22,D26,D66
      READ(5,*) F11,F12,F16,F22,F26,F66
      READ(5,*) H11,H12,H16,H22,H26,H66
C
C     INTRODUCE SHEAR CORRECTION TERMS (FOR FIRST ORDER THEORIES)
      IF(HOT.EQ.1) GOTO 5
      IF(K1.EQ.0.0) THEN
C       SINGLE COEFFICIENT
        A44=K*A44
        A55=K*A55
      ELSE
C       DOUBLE COEFFICIENT
        A44=K2*A44
        A55=K1*A55
      ENDIF
C
C     CALCULATE ASPECT RATIO, SPAN-TO-DEPTH RATIO, AND MASS DENSITY
5     R=A/B
      S=A/H
      P1=RHO*H
      P1=P1/(32.1740*12.0)
C
C     FORM NESTED LOOPS TO GENERATE GALERKIN TERMS
      I=1
      J=1
      DO 10 P=1,MMAX
      DO 10 Q=1,NMAX
      DO 20 M=1,MMAX
      DO 20 N=1,NMAX
C
```

```
C     PERFORM INTEGRATION
C
      EOTEST1=MOD(M+P,2)
      EOTEST2=MOD(N+Q,2)
      IF(M.EQ.P) THEN
        F1=0.5
        F3=0.0
        F5=0.0
      ELSE
        F1=0.0
        IF(EOTEST1.EQ.0) THEN
          F3=0.0
          F5=0.0
        ELSE
          F3=2.0*P/(PI*(P**2-M**2))
          F5=2.0*M/(PI*(M**2-P**2))
        ENDIF
      ENDIF
      IF(N.EQ.Q) THEN
        F2=0.5
        F4=0.0
        F6=0.0
      ELSE
        F2=0.0
        IF(EOTEST2.EQ.0) THEN
          F4=0.0
          F6=0.0
        ELSE
          F4=2.0*Q/(PI*(Q**2-N**2))
          F6=2.0*N/(PI*(N**2-Q**2))
        ENDIF
      ENDIF
C
C     COMPUTE STIFFNESS TERMS
C
      A1=F1*F2*(S**2*M**2*PI2)*(-D11+8./3.*F11-16./9.*H11)+F1*F2*(S**2*R
     +**2*N**2*PI2)*(-D66+8./3.*F66-16./9.*H66)+F1*F2*(S**4)*(-A55+8.*D5
     +5-16.*F55)+F3*F4*(S**2*R*M*N*PI2)*(2.*D16-16./3.*F16+32./9.*H16)
C
      B1=F3*F4*(S**2*R*M*N*PI2)*(+D12-8./3.*F12+D66-8./3.*F66+16./9.*H12
     ++16./9.*H66)+F1*F2*(S**2*M**2*PI2)*(-D16+8./3.*F16-16./9.*H16)+F1*
     +F2*(S**4)*(-A45+8.*D45-16.*F45)+F1*F2*(S**2*R**2*N**2*PI2)*(-D26+8
     +./3.*F26-16./9.*H26)
C
      C1=F3*F2*(S*M**3*PI3)*(4./3.*F11-16./9.*H11)+F3*F2*(S*R**2*M*N**2*
     +PI3)*(4./3.*F12+8./3.*F66-16./9.*H12-32./9.*H66)+F3*F2*(S**3*M*PI)
     +*(-A55+8.*D55-16.*F55)+F1*F4*(S*R*M**2*N*PI3)*(4.*F16-48./9.*H16)+
     +F1*F4*(S*R**3*N**3*PI3)*(4./3.*F26-16./9.*H26)+F1*F4*(S**3*R*N*PI)
     +*(-A45+8.*D45-16.*F45)
C
      A2=F3*F4*(S**2*R*M*N*PI2)*(+D12-8./3.*F12+D66-4./3.*F26-8./3.*F66+
     +16./9.*H12+16./9.*H66)+F1*F2*(S**2*R**2*N**2*PI2)*(-D26+8./3.*F26-
     +16./9.*H26)+F1*F2*(S**4)*(-A45+8.*D45-16.*F45)+F1*F2*(S**2*M**2*PI
     +2)*(-D16+8./3.*F16-16./9.*H16)
C
```

```
      B2=F3*F4*(S**2*R*M*N*PI2)*(2.*D26-4.*F26+32./9.*H26)+F1*F2*(S**2*R
     +**2*N**2*PI2)*(-D22+8./3.*F22-16./9.*H22)+F1*F2*(S**2*M**2*PI2)*(-
     +D66+8./3.*F66-16./9.*H66)+F1*F2*(S**4)*(-A44+8.*D44-16.*F44)
C
      C2=F1*F4*(S*R*M**2*N*PI3)*(4./3.*F12+8./3.*F66-16./9.*H12-32./9.*H
     +66)+F1*F4*(S*R**3*N**3*PI3)*(4./3.*F22-16./9.*H22)+F1*F4*(S**3*R*N
     +*PI)*(-A44+8.*D44-16.*F44)+F3*F2*(S*R**2*M*N**2*PI3)*(4.*F26-48./9
     +.*H26)+F3*F2*(S*M**3*PI3)*(4./3.*F16-16./9.*H16)+F3*F2*(S**3*M*PI)
     +*(-A45+8.*D45-16.*F45)
C
      A3=F3*F2*(S**3*M*PI)*(A55-8.*D55+16.*F55)+F3*F2*(S*M**3*PI3)*(-4./
     +3.*F11+16./9.*H11)+F3*F2*(S*R**2*M*N**2*PI3)*(-4./3.*F12+16./9.*H1
     +2-8./3.*F66+32./9.*H66)+F1*F4*(S**3*R*N*PI)*(A45-8.*D45+16.*F45)+F
     +1*F4*(S*R*M**2*N*PI3)*(-4.*F16+48./9.*H16)+F1*F1*(S*R**3*N**3*PI3)
     +*(-4./3.*F26+16./9.*H26)
C
      B3=F1*F4*(S**3*R*N*PI)*(+A44-8.*D44+16.*F44)+F1*F4*(S*R*M**2*N*PI3
     +)*(-4./3.*F12+16./9.*H12-8./3.*F66+32./9.*H66)+F1*F4*(S*R**3*N**3*
     +PI3)*(-4./3.*F22+16./9.*H22)+F3*F2*(S**3*M*PI)*(A45-8.*D45+16.*F45
     +)+F3*F2*(S*M**3*PI3)*(-4./3.*F16+16./9.*H16)+F3*F2*(S*R**2*M*N**2*
     +PI3)*(-4.*F26+48./9.*H26)
C
      C3=F1*F2*(S**2*M**2*PI2)*(-A55+8.*D55-16.*F55)+F1*F2*(S**2*R**2*N*
     +*2*PI2)*(-A44+8.*D44-16.*F44)+F1*F2*(PI4)*(-16./9.*H11*M**4-32./9.
     +*H12*R**2*M**2*N**2-16./9.*H22*R**4*N**4-64./9.*H66*R**2*M**2*N**2
     +)+F3*F4*(2.*S**2*R*M*N*PI2)*(A45-8.*D45+16.*F45)+F3*F4*(64./9.*PI4
     +)*(H16*R*M**3*N+H26*R**3*M*N**3)
C
C     STORE STIFFNESS TERMS IN ST MATRIX
C
      ST(I,J)=-A1
      ST(I,J+MMAX*NMAX)=-B1
      ST(I,J+2*MMAX*NMAX)=-C1
      ST(I+MMAX*NMAX,J)=-A2
      ST(I+MMAX*NMAX,J+MMAX*NMAX)=-B2
      ST(I+MMAX*NMAX,J+2*MMAX*NMAX)=-C2
      ST(I+2*MMAX*NMAX,J)=-A3
      ST(I+2*MMAX*NMAX,J+MMAX*NMAX)=-B3
      ST(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=-C3
C
C     COMPUTE MASS/INERTIA TERMS
C
      IF(VIB.EQ.1) THEN
C       VIBRATION MASS/INERTIA TERMS
        IF(HOT.EQ.1) THEN
C         HIGHER ORDER THEORY
          LAM1=A**2*PI/(ET*H**3)
          X1=F1*F2*(17./315.)*(S**2)*LAM1
          Y1=0.0
          Z1=F3*F2*(-4./315.)*(S*M*PI)*LAM1
          X2=0.0
          Y2=F1*F2*(17./315.)*(S**2)*LAM1
          Z2=F1*F4*(-4./315.)*(S*R*N*PI)*LAM1
          X3=F3*F2*(4./315.)*(S*M*PI)*LAM1
          Y3=F1*F4*(4./315.)*(S*R*N*PI)*LAM1
```

```
            Z3=F1*F2*(S**2+16./4032.*(M**2*PI2)+16./4032.*(R**2*N**2*PI2))
     +*LAM1
          ELSE
C           FIRST ORDER THEORY
            X1=ROT*F1*F2*(A**4*P1/(12.*ET*H**3))
            Y1=0.0
            Z1=0.0
            X2=0.0
            Y2=ROT*F1*F2*(A**4*P1/(12.*ET*H**3))
            Z2=0.0
            X3=0.0
            Y3=0.0
            Z3=F1*F2*(A**4*P1/(ET*H**4))
          ENDIF
        ELSE
C         BUCKLING MASS/INERTIA TERMS (IDENTICAL FOR ALL THEORIES)
          LAM2=A**2/(ET*H**3)
          X1=0.0
          Y1=0.0
          Z1=0.0
          X2=0.0
          Y2=0.0
          Z2=0.0
          X3=0.0
          Y3=0.0
          Z3=F1*F2*(KK1*M**2*PI2+R**2*KK2*N**2*PI2)*LAM2+F3*F4*(2.*R*KK3*
     +M*N*PI2)*LAM2
        ENDIF
C
C       STORE MASS/INERTIA TERMS IN MA MATRIX
C
          MA(I,J)=X1
          MA(I,J+MMAX*NMAX)=Y1
          MA(I,J+2*MMAX*NMAX)=Z1
          MA(I+MMAX*NMAX,J)=X2
          MA(I+MMAX*NMAX,J+MMAX*NMAX)=Y2
          MA(I+MMAX*NMAX,J+2*MMAX*NMAX)=Z2
          MA(I+2*MMAX*NMAX,J)=X3
          MA(I+2*MMAX*NMAX,J+MMAX*NMAX)=Y3
          MA(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=Z3
C
        J=J+1
20      CONTINUE
        I=I+1
        J=1
10      CONTINUE
C
C       WRITE STIFFNESS AND MASS/INERTIA MATRICIES TO A FILE
        WRITE(6,45) MMAX
45      FORMAT(I2)
        DO 40 I=1,3*MMAX*NMAX
        DO 40 J=1,3*MMAX*NMAX
        WRITE(6,50) ST(I,J),MA(I,J)
50      FORMAT(2E16.8)
40      CONTINUE
```

```
STOP
END
```

```
      PROGRAM GALERK3
C     CLAMPED - SIMPLY SUPPORTED BOUNDARIES
      DIMENSION ST(300,300)
      REAL K,K1,K2,MA(300,300),KK1,KK2,KK3,LAM1,LAM2
      INTEGER M,N,P,Q,EOTEST1,EOTEST2,MMAX,NMAX,HOT,VIB
      OPEN(5,FILE='GALIN')
      OPEN(6,FILE='EIGIN')
C
C     DEFINE CONSTANTS
      K=5.0/6.0
      PI=3.1415926535
      PI2=PI**2
      PI3=PI**3
      PI4=PI**4
C
C     READ INPUT DATA
C     HIGHER ORDER THEORY, VIBRATION, AND ROTATORY INERTIA FLAGS
      READ(5,*) HOT,VIB,ROT
      READ(5,*) MMAX,NMAX
      READ(5,*) A,B,H
      READ(5,*) RHO,ET,K1,K2,KK1,KK2,KK3
      READ(5,*) A44,A45,A55
      READ(5,*) D44,D45,D55
      READ(5,*) F44,F45,F55
      READ(5,*) D11,D12,D16,D22,D26,D66
      READ(5,*) F11,F12,F16,F22,F26,F66
      READ(5,*) H11,H12,H16,H22,H26,H66
C
C     INTRODUCE SHEAR CORRECTION TERMS (FOR FIRST ORDER THEORIES)
      IF(HOT.EQ.1) GOTO 5
      IF(K1.EQ.0.0) THEN
C       SINGLE COEFFICIENT
        A44=K*A44
        A55=K*A55
      ELSE
C       DOUBLE COEFFICIENT
        A44=K2*A44
        A55=K1*A55
      ENDIF
C
C     CALCULATE ASPECT RATIO, SPAN-TO-DEPTH RATIO, AND MASS DENSITY
5     R=A/B
      S=A/H
      P1=RHO*H
      P1=P1/(32.1740*12.0)
C
C     FORM NESTED LOOPS TO GENERATE GALERKIN TERMS
      I=1
      J=1
      DO 10 P=1,MMAX
      DO 10 Q=1,NMAX
      DO 20 M=1,MMAX
      DO 20 N=1,NMAX
C
C     PERFORM INTEGRATION
```

```
      EOTEST1=MOD(M+P,2)
      EOTEST2=MOD(N+Q,2)
      IF(M.EQ.P) THEN
        F1=0.5
        F3=0.0
        F5=0.0
      ELSE
        F1=0.0
        IF(EOTEST1.EQ.0) THEN
          F3=0.0
          F5=0.0
        ELSE
          F3=2.0*P/(PI*(P**2-M**2))
          F5=2.0*M/(PI*(M**2-P**2))
        ENDIF
      ENDIF
      IF(N.EQ.Q) THEN
        F2=0.5
        F4=0.0
        F6=0.0
      ELSE
        F2=0.0
        IF(EOTEST2.EQ.0) THEN
          F4=0.0
          F6=0.0
        ELSE
          F4=2.0*Q/(PI*(Q**2-N**2))
          F6=2.0*N/(PI*(N**2-Q**2))
        ENDIF
      ENDIF
C
C     COMPUTE STIFFNESS TERMS
      A1=F1*F2*(S**2*M**2*PI2)*(-D11+8./3.*F11-16./9.*H11)+F1*F2*(S**2*R
     +**2*N**2*PI2)*(-D66+8./3.*F66-16./9.*H66)+F1*F2*(S**4)*(-A55+8.*D5
     +5-16.*F55)+F3*F4*(S**2*R*M*N*PI2)*(2.*D16-16./3.*F16+32./9.*H16)
C
      B1=F3*F2*(S**2*R*M*N*PI2)*(-D12+8./3.*F12-D66+8./3.*F66-16./9.*H12
     +-16./9.*H66)+F1*F4*(S**2*M**2*PI2)*(-D16+8./3.*F16-16./9.*H16)+F1*
     +F4*(S**4)*(-A45+8.*D45-16.*F45)+F1*F4*(S**2*R**2*N**2*PI2)*(-D26+8
     +./3.*F26-16./9.*H26)
C
      C1=F3*F2*(S*M**3*PI3)*(4./3.*F11-16./9.*H11)+F3*F2*(S*R**2*M*N**2*
     +PI3)*(4./3.*F12+8./3.*F66-16./9.*H12-32./9.*H66)+F3*F2*(S**3*M*PI)
     +*(-A55+8.*D55-16.*F55)+F1*F4*(S*R*M**2*N*PI3)*(4.*F16-48./9.*H16)+
     +F1*F4*(S*R**3*N**3*PI3)*(4./3.*F26-16./9.*H26)+F1*F4*(S**3*R*N*PI)
     +*(-A45+8.*D45-16.*F45)
C
      A2=F3*F2*(S**2*R*M*N*PI2)*(+D12-8./3.*F12+D66-4./3.*F26-8./3.*F66+
     +16./9.*H12+16./9.*H66)+F1*F6*(S**2*R**2*N**2*PI2)*(-D26+8./3.*F26-
     +16./9.*H26)+F1*F6*(S**4)*(-A45+8.*D45-16.*F45)+F1*F6*(S**2*M**2*PI
     +2)*(-D16+8./3.*F16-16./9.*H16)+F1*(1.-COS(N*PI)*COS(Q*PI))*(S**2*R
     +*N*PI)*(D26-4.*F26+32./9.*H26)
C
      B2=F1*F2*(S**2*R**2*N**2*PI2)*(-D22+8./3.*F22-16./9.*H22)+F1*F2*(S
     +**2*M**2*PI2)*(-D66+8./3.*F66-16./9.*H66)+F1*F2*(S**4)*(-A44+8.*D4
```

```
     +4-16.*F44)+F3*F6*(S**2*R*M*N*PI2)*(-2.*D26+4.*F26-32./9.*H26)+F3*(
     +1.-COS(N*PI)*COS(Q*PI))*(S**2*M*PI)*(D26-4.*F26+32./9.*H26)
C
      C2=F1*F2*(S*R*M**2*N*PI3)*(4./3.*F12+8./3.*F66-16./9.*H12-32./9.*H
     +66)+F1*F2*(S*R**3*N**3*PI3)*(4./3.*F22-16./9.*H22)+F1*F2*(S**3*R*N
     +*PI)*(-A44+8.*D44-16.*F44)+F3*F6*(S*R**2*M*N**2*PI3)*(4.*F26-48./9
     +.*H26)+F3*F6*(S**3*M*PI)*(-A45+8.*D45-16.*F45)+F3*F6*(S*M**3*PI3)*
     +(4./3.*F16-16./9.*H16)+F3*(1.-COS(N*PI)*COS(Q*PI))*(S*R*M*N*PI2)*(
     +-8./3.*F26+64./9.*H26)
C
      A3=F3*F2*(S**3*M*PI)*(+A55-8.*D55+16.*F55)+F3*F2*(S*M**3*PI3)*(-4.
     +/3.*F11+16./9.*H11)+F3*F2*(S*R**2*M*N**2*PI3)*(-4./3.*F12+16./9.*H
     +12-8./3.*F66+32./9.*H66)+F1*F4*(S**3*R*N*PI)*(A45-8.*D45+16.*F45)+
     +F1*F4*(S*R*M**2*N*PI3)*(-4.*F16+48./9.*H16)+F1*F4*(S*R**3*N**3*PI3
     +)*(-4./3.*F26+16./9.*H26)
C
      B3=F1*F2*(S**3*R*N*PI)*(-A44+8.*D44-16.*F44)+F1*F2*(S*R*M**2*N*PI3
     +)*(4./3.*F12-16./9.*H12+8./3.*F66-32./9.*H66)+F1*F2*(S*R**3*N**3*P
     +I3)*(4./3.*F22-16./9.*H22)+F3*F4*(S*M**3*PI3)*(-4./3.*F16+16./9.*H
     +16)+F3*F4*(S**3*M*PI)*(A45-8.*D45+16.*F45)+F3*F4*(S*R**2*M*N**2*PI
     +3)*(-4.*F26+48./9.*H26)
C
      C3=F1*F2*(S**2*M**2*PI2)*(-A55+8.*D55-16.*F55)+F1*F2*(S**2*R**2*N*
     +*2*PI2)*(-A44+8.*D44-16.*F44)+F1*F2*(R**2*M**2*N**2*PI4)*(-32./9.*
     +H12-64./9.*H66)+F1*F2*PI4*(-16./9.*H11*M**4-16./9.*H22*R**4*N**4)+
     +F3*F4*(2.*S**2*R*M*N*PI2)*(A45-8.*D45+16.*F45)+F3*F4*PI4*(64./9.*H
     +16*R*M**3*N+64./9.*H26*R**3*M*N**3)
C
C     STORE STIFFNESS TERMS IN ST MATRIX
      ST(I,J)=-A1
      ST(I,J+MMAX*NMAX)=-B1
      ST(I,J+2*MMAX*NMAX)=-C1
      ST(I+MMAX*NMAX,J)=-A2
      ST(I+MMAX*NMAX,J+MMAX*NMAX)=-B2
      ST(I+MMAX*NMAX,J+2*MMAX*NMAX)=-C2
      ST(I+2*MMAX*NMAX,J)=-A3
      ST(I+2*MMAX*NMAX,J+MMAX*NMAX)=-B3
      ST(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=-C3
C
C     COMPUTE MASS/INERTIA TERMS
      IF(VIB.EQ.1) THEN
C       VIBRATION MASS/INERTIA TERMS
        IF(HOT.EQ.1) THEN
C         HIGHER ORDER THEORY
          LAM1=A**2*P1/(ET*H**3)
          X1=F1*F2*(17./315.)*(S**2)*LAM1
          Y1=0.0
          Z1=F3*F2*(-4./315.)*(S*M*PI)*LAM1
          X2=0.0
          Y2=F1*F2*(17./315.)*(S**2)*LAM1
          Z2=F1*F2*(-4./315.)*(S*R*N*PI)*LAM1
          X3=F3*F2*(4./315.)*(S*M*PI)*LAM1
          Y3=F1*F2*(-4./315.)*(S*R*N*PI)*LAM1
          Z3=F1*F2*(S**2+16./4032.*(M**2*PI2)+16./4032.*(R**2*N**2*PI2))
     +*LAM1
```

```
      ELSE
C         FIRST ORDER THEORY
          X1=ROT*F1*F2*(A**4*P1/(12.*ET*H**3))
          Y1=0.0
          Z1=0.0
          X2=0.0
          Y2=ROT*F1*F2*(A**4*P1/(12.*ET*H**3))
          Z2=0.0
          X3=0.0
          Y3=0.0
          Z3=F1*F2*(A**4*P1/(ET*H**4))
        ENDIF
      ELSE
C       BUCKLING MASS/INERTIA TERMS (IDENTICAL FOR ALL THEORIES)
        LAM2=A**2/(ET*H**3)
        X1=0.0
        Y1=0.0
        Z1=0.0
        X2=0.0
        Y2=0.0
        Z2=0.0
        X3=0.0
        Y3=0.0
        Z3=F1*F2*(KK1*M**2*PI2+R**2*KK2*N**2*PI2)*LAM2+F3*F4*(2.*R*KK3*
     +M*N*PI2)*LAM2
      ENDIF
C
C     STORE MASS/INERTIA TERMS IN MA MATRIX
        MA(I,J)=X1
        MA(I,J+MMAX*NMAX)=Y1
        MA(I,J+2*MMAX*NMAX)=Z1
        MA(I+MMAX*NMAX,J)=X2
        MA(I+MMAX*NMAX,J+MMAX*NMAX)=Y2
        MA(I+MMAX*NMAX,J+2*MMAX*NMAX)=Z2
        MA(I+2*MMAX*NMAX,J)=X3
        MA(I+2*MMAX*NMAX,J+MMAX*NMAX)=Y3
        MA(I+2*MMAX*NMAX,J+2*MMAX*NMAX)=Z3
C
      J=J+1
20    CONTINUE
      I=I+1
      J=1
10    CONTINUE
C
C     WRITE STIFFNESS AND MASS/INERTIA MATRICIES TO A FILE
      WRITE(6,45) MMAX
45    FORMAT(I2)
      DO 40 I=1,3*MMAX*NMAX
      DO 40 J=1,3*MMAX*NMAX
      WRITE(6,50) ST(I,J),MA(I,J)
50    FORMAT(2E16.8)
40    CONTINUE
      STOP
      END
```

Appendix C

```
      PROGRAM EIGEN
      REAL MA(300,300),RWKSP(300000)
      DIMENSION ST(300,300),A(90000),B(90000),BETA(300)
      COMPLEX ALPHA(300),EVAL(300),EVEC(300,300)
      INTEGER I,J,MMAX,N,LDA,LDB,LDEVEC
      EXTERNAL GVCRG,GPIRG
      COMMON/WORKSP/ RWKSP
      CALL IWKIN(300000)
C
      OPEN(5,FILE='EIGIN')
      OPEN(7,FILE='EIGOUT')
C
C     LOAD STIFFNESS AND MASS/INERTIA MATRICIES
      READ(5,5) MMAX
5     FORMAT(I2)
      N=3*MMAX*MMAX
C
      K=1
      DO 10 I=1,N
      DO 10 J=1,N
      READ(5,*) ST(J,I),MA(J,I)
      A(K)=ST(J,I)
      B(K)=MA(J,I)
      K=K+1
10    CONTINUE
C
      LDA=N
      LDB=N
      LDEVEC=N
C
C     COMPUTE EIGENVALUES AND EIGENVECTORS
      CALL GVCRG(N,A,LDA,B,LDB,ALPHA,BETA,EVEC,LDEVEC)
C
C     COMPUTE PERFORMANCE INDEX
      PI=GPIRG(N,N,A,LDA,B,LDB,ALPHA,BETA,EVEC,LDEVEC)
C
C     PRINT RESULTS
      WRITE(7,80) PI
80    FORMAT('PERFORMANCE INDEX = ',F10.3)
      DO 40 I=1,N
      IF (BETA(I).NE.0.0) THEN
        EVAL(I)=ALPHA(I)/BETA(I)
      ELSE
C     INFINITE EIGENVALUE
        EVAL(I)=AMACH(2)
      ENDIF
C
      WRITE(7,50) I,EVAL(I)
50    FORMAT('EVAL(',I3,')=',E16.8,5X,E16.8)
40    CONTINUE
C
C     DO 60 I=1,N
C     DO 60 J=1,N
C     WRITE(7,70) I,J,EVEC(I,J)
70    FORMAT('EVEC(',I3,1X,I3,')=',E16.8,5X,E16.8)
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```
60    CONTINUE
      STOP
      END
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This work was initiated by a need for solutions to a higher order shear deformation plate theory, which better approximates the through-the-thickness deformation and interlaminar shear stresses, using plates other than simply supported. The non-simply supported boundary conditons necessitated the Galerkin approximate technique. Solutions desired were natural frequencies and buckling loads for the following boundary conditions: simply supported, clamped, and clamped-simply supported. Two graphite/epoxy laminates of different construction were investigated for effects of varying span-to-depth ratios. Comparisons of the higher order theory with linear and classical theory were made along with convergence characteristics of the Galerkin technique. Results of the work showed that the higher order theory predicted natural frequencies and buckling loads about 6 percent lower than the first order theory. The higher order theory compared well with classical laminated plate theory for thin plates. Convergence characteristics of the Galerkin technique was excellent overall.